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Chromatic coupling induced by skew sextupolar field errors in the LHC main dipoles and its correction

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Abstract

In a recent study dealing with the chromatic behaviour of the LHC in the presence of multi-polar field errors, it has been clearly shown that the skew sextupolar systematic component of the arc dipoles was liable to generate a spectacular second order chromaticity, up to 50 times higher than its tolerance value estimated at $Q'' = 1000$ at injection. This result was qualitatively explained by transverse coupling phenomena induced in the arcs (dispersive regions) and affecting the dynamics of off-momentum particles (linear coupling proportional to $\delta p/p$ or 'chromatic coupling'). This paper presents an analytical approach of the problem based on the canonical perturbation theory and explaining perfectly the chromatic coupling phenomena induced in the LHC. This understanding is used to design a simple and powerful a_3 compensation scheme which consists in replacing in each arc two pairs of lattice (normal) sextupoles by two pairs of skew sextupoles judiciously positioned. Tracking results performed on the LHC lattice version 6 illustrate the beneficial impact of the correction on the dynamic aperture of the ring.

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Introduction

The dynamic aperture of the LHC is limited by strong geometric aberrations induced by the multi-polar components of its super-conducting magnets, especially at injection energy. In order to face this problem, many efforts have been already done by adding non-linear correctors in the vicinity of the most sensitive components: (among others) a_6 and b_6 spool pieces in the inner triplets, b_3, b_4, b_5 and possibly a_4 correctors at the ends of the bending magnets. In a recent study [1], it has been shown that the skew sextupolar systematic component of the main dipoles could also affect significantly the stability of the motion at injection. The signature of this effect can be a spectacular second order chromaticity Q'' generated in the arcs by linear coupling phenomena proportional to $\delta p/p$ (or chromatic coupling); one of the main conclusions of this study was the necessity of designing a dedicated a_3 correction scheme.

As we will see, the main source of the component a_3 in the LHC is of geometric origin in the main dipoles, systematic arc by arc with an average value by arc in between ± 0.867 (in units of $10^{\ 4}$ and measured at the reference radius $R_r = 17$ mm). This value corresponds to an integrated strength per arc

$$\left(K_2\ L\right)_{arc} = 2\ \frac{(B_0 L)_{arc}}{(B\rho)}\ \frac{a_3}{R_r^2} = \frac{4\pi}{8}\ \frac{a_3}{R_r^2} \sim 0.47\,\mathrm{m}^{\ 2}\ ,$$

roughly equal to that of the horizontal chromaticity sextupoles MSCH at injection. This "pseudo-comparison" gives an idea of the extent of the perturbation.

We will begin by defining some acceptability criteria concerning the chromatic aberrations of the LHC both at the injection and collision energies (Chapter 1). The latter will be obtained, on the one hand, by constraining the maximum tune spread induced in the on-momentum beam by the chromatic detuning; on the other hand, they will be chosen according to the ability to accelerate/decelerate safely the beam in a relevant momentum range for chromaticity measurements. These criteria will then be expressed in terms of tolerances on Q'' at injection and collision, to be compared with the second order chromaticities that may be induced by chromatic coupling. In conclusion, we will show that a dedicated a_3 correction scheme is imperative in the LHC and that the correction must be ensured up to the collision energy.

The orders of magnitude announced previously and concerning the effect on Q'' induced by the skew sextupolar field errors will be obtained in a rigorous way in the second chapter. The problem will be completely clarified by an analytical treatment using the canonical perturbation theory (Section 2.1 & 2.2). In particular, the expressions of the second order chromaticities Q''_I and Q''_{II} will be derived in an integral form as a function of the azimuthal distribution $a_3(s)$. These expressions will be compared to the ones predicted by the resonance theory.

By inspecting the possible different sources of a_3 in the LHC, we will show that the one of geometric origin in the main dipoles is indisputably the most critical. Then, we will be able to complete the exact computation of the second order chromaticities induced (Section 2.3). The spectacular orders of magnitude obtained for Q'' in some particular

case depending on the tune split will demonstrate that a dedicated correction scheme is imperative.
Simulations with MAD [2] performed on LHC Version 6.-2 (for two different tune splits, 4 and 5) will validate the analytical results obtained (Section 2.4).

By analysing all the terms occurring in the expressions of $Q''_{I,II}$ previously calculated, some decoupling criteria will be revealed (Section 3.1), then partially or fully used in the different correction schemes which will be tested.
The a_3 correction scheme presently proposed in the LHC lattice version 6.-2 will be firstly discussed (Section 3.2). The latter consists in replacing in each arc six lattice octupoles by a family of six skew sextupoles of 32 cm each (that is a total of 96 skew sextupoles for the two rings). For many reasons (insufficient available gradient, geometric aberrations and then amplitude detuning induced, poor quality of the correction), this scheme will be given up and other options will be studied.
The first two ones will pursue the idea of using the existing lattice components in order to perform the correction:

- by using the vertical orbit correctors of the arcs in order to generate vertical orbit deviations (vertical bumps) within the arc octupoles (Section 3.3): as a result, a skew sextupolar component will be created at the level of these octupoles. For reasons of mechanical aperture, this option will be immediately rejected.

- by generating vertical dispersion bumps at the level of some lattice sextupoles, the strength K_2^+ of which will be at least 10 times higher than the one of the chromatic correction sextupoles (Section 3.4). As a result, off-momentum particles will see a skew quadrupolar field error a_2 proportional to $K_2^+ \, D_y \, \delta$ (where D_y is the vertical dispersion induced) which will be susceptible to compensate the chromatic coupling generated by the component a_3 of dipoles. These vertical dispersion bumps will be generated, either by vertical orbit correctors, or by using the arc skew quadrupoles dedicated to the a_2 correction and judiciously re-positioned for our purpose. For reasons of mechanical aperture and available strength, these options will also have to be rejected.

Finally, an elegant solution will be presented in detail and proposed for the a_3 correction in the LHC lattice version 6.-2 (Section 3.5). As we will see, this correction scheme will precisely respect the different decoupling criteria previously mentioned. It simply consists in removing from each arc of the two rings two pairs of lattice sextupoles and in replacing them by two pairs of skew sextupoles of same length. The advantage of this scheme lies in the fact that it does not require the creation of a new type of magnet. Indeed, as for the chromatic correction sextupoles, these skew sextupoles must be combined with an horizontal or vertical orbit corrector depending on whether they will be placed close to a QF (x-focusing quadrupole of MQ type) or close to a QD. By naming these skew sextupoles SSF and SSD respectively, it is easy to see that a chromatic correction sextupole MSCV (normal sextupole combined with a vertical corrector) becomes a skew sextupole of SSF type by a simple mechanical rotation of 90 and, in the same way, that a skew sextupole SSD can be obtained by tilting a chromaticity sextupole MSCH. The gradient increase of the remaining chromatic correction sextupoles will be discussed by referring to a previous study [3]. Due to the fact that the safety margin on the strength of the MSCV's is relatively small at the collision energy, we will favour a solution for which only skew sextupoles of SSF type are involved, judiciously positioned at the place of some lattice sextupoles MSCH. Finally, unlike the previous schemes, the correctors SSF will be strong enough to ensure the a_3 compensation up to the collision energy and even more.

In the last chapter (Chapter 4), 6-D tracking results will illustrate the beneficial impact of this correction on the LHC dynamic aperture, and then, a fortiori, will check that the scheme proposed does not excite any higher order effects.

1 Criteria for chromatic aberrations

1.1 The momentum window at injection and collision

We begin by defining the momentum range over which the optical functions of the LHC, mainly the tune dependence on momentum $Q(\delta)$, shall be well behaved. According to Ref. [1], we have to consider requirements of different natures:

- (i) the natural momentum spread in the beam. At injection, the total relative momentum spread of the beam is about $2\,\sigma_\delta \sim \mathbf{10^{\;\,3}}$ (full width) and, at collision, of the order of $\mathbf{2.5\,10^{\;\,4}}$. A small chromatic detuning over this range is a minimum requirement from the beam dynamics point of view.

- (ii) the ability to measure the linear chromaticity, both at injection and collision (and during the ramp). For the LHC, the nominal chromaticity is chosen to be 2 units; it has to be positive to avoid head-tail instability and less than 5 units according to tracking results showing that the total tune spread (chromatic and amplitude detuning) must not exceed $5\,10^{\;\,3}$ at injection. It is estimated that a good accuracy on the linear chromaticity measurement is reached by fitting the points obtained in a window defined by $\delta = \pm\mathbf{5\,10^{\;\,4}}$. Indeed, at both extremities of this window, the tune shift induced by a Q' of 2 units is equal to $10^{\;\,3}$, which is sufficient assuming a noise level of $10^{\;\,4}$ in tune.

- (iii) the understanding of non-linearities. The measurement of the non-linear chromaticities is the simplest way to check the correction quality of higher-order multipoles. The case of b_5 is a good example to be considered. Indeed in the LHC, the multipole b_5, if not well corrected, is large enough to be detrimental to the dynamic aperture. Among the different effects that it induces, the following two can be retained: a third order chromaticity Q''' and a first order amplitude detuning proportional to $\delta p/p$ due to octupolar field errors seen by off-momentum particles. The second effect is found to be dominant but is more delicate to measure. Nevertheless, insofar as the correction of the b_5 component of dipoles is locally achieved (thanks to b_5 spool pieces attached to each arc dipole), the latter is strongly minimised if the observable Q''' is put back to zero. In order to reach a good enough accuracy on this kind of measurement, required at least at injection, it is estimated that the momentum window must be enlarged up to $\delta = \pm\mathbf{2\,10^{\;\,3}}$ [1].

1.2 Tolerances on Q'' at injection and collision

We start from the simple following equation

$$Q = Q_0 + Q'\delta + \frac{1}{2}Q''\delta^2 + \dots \ , \tag{1}$$

and deduce from the previous section the tolerances on the second order chromaticity required both at the injection and collision energies.

- *Linear chromaticity measurement.* In order to avoid a fast head-tail instability during the measurement of Q', the function $Q'(\delta) = Q' + Q''\delta$ must remain positive in the required momentum window $\delta = \pm 5\,10^{\;\;4}$. This measurement shall be done both at injection and at collision. In both cases, the linear chromaticity of the LHC being chosen to be 2 units, we obtain a first constraint on Q'':

$$Q'(\delta) = 2 + Q''\delta > 0 \ \text{ for } \ \delta = \pm 5\,10^{\;\;4} \Rightarrow |Q''| < 4000\,.$$

- *Non-linear chromaticity measurement.* For non-linear chromaticity measurements, at least at injection, the previous criteria must be satisfied in a largest window corresponding to $\delta = \pm 2\,10^{\;\;3}$. The tolerance on Q'' at injection is then tighter:

$$|Q''| < 1000 \ \text{ at injection.}$$

- *On-momentum beam.* If these tolerances are respected, let us now check that the average chromatic detuning induced in the on-momentum beam, $\Delta Q_m = Q''\sigma_\delta^2/2$, and the tune ripple sampled by the tail of the bunch,

$$\Delta Q(s) = Q''/2 \times \left(2\,\sigma_\delta\,\cos(2\pi\,Q_s\,s/C)\right)^2 = 2\,\Delta Q_m\left(1 + \cos(4\pi\,Q_s\,s/C)\right)$$

  (where Q_s denotes the synchrotron tune and C is the ring circumference), are relatively small, say $\Delta Q_{max} \equiv 4|\Delta Q_m| < 10^{\;\;3}$. That is effectively the case since $|\Delta Q_m| < 1.25\,10^{\;\;4}$ at injection (for $|Q''| < 1000$), and $|\Delta Q_m| < 3\,10^{\;\;5}$ at the top energy (for $|Q''| < 4000$).

The specifications on the second order chromaticity are then:

$$\begin{cases} |Q''| & < \ 1000 \ \text{ at injection} \\ |Q''| & < \ 4000 \ \text{ at collision (and during the ramp).} \end{cases} \tag{2}$$

Depending on the choice of the working point, we will see in the next chapter that second order chromaticities of about 57000 units (for a tune split of 4) or 13000 units (for a tune split of 5) may be induced at injection by chromatic coupling phenomena (see Section 2.4). These values concern an optics for which the distance Δ to the closest difference resonance $Q_x{-}Q_y{-}p{=}0$ is equal to 0.03 (by referring to the nominal injection optics of the LHC version 6.-2) and scales roughly as a_3^2/Δ where a_3 is related to the skew sextupolar field errors of the dipoles. Knowing that the distance Δ is reduced to 0.01 for the nominal collision optics and that the component a_3 of geometric origin is independent of the energy, the values of Q'' previously announced have to be multiplied by a factor of 3 at the top energy, and possibly by a factor of 10, since operation at $\Delta = 0.003$ would be advantageous in collision.

Under these conditions and according to the tolerances on Q'' given previously, a full correction of the a_3 is then imperative up to the collision energy.

2 Approach of the problem by the canonical perturbation theory

The purpose of this chapter is to derive an analytical expression for the second order chromaticity generated by a given distribution $K_2\,(s) = 2/\rho\; a_3(s)/R_r^2$ of skew sextupolar field errors along a ring and to apply the results obtained to the LHC.
Let us begin the discussion with the following simple remark. An horizontal displacement δx in a skew sextupole of strength K_2 is analogous to a vertical orbit deviation in a normal sextupole: a skew quadrupole arises, with the strength $K_1 = K_2 * \delta x$. Therefore, an off-momentum particle of energy $p_0(1+\delta)$ will see a skew quadrupolar field of strength

$$K_1\,(s) = K_2\,(s)\,D_x(s)\,\frac{\delta}{1+\delta} \sim K_2\,(s)\,D_x(s)\,\delta\ , \tag{3}$$

where $D_x(s)$ denotes the horizontal dispersion function. The calculation of the second order chromaticities $Q''_{I,II}$ induced by the distribution $K_2\,(s)$ is then equivalent to the following problem: given a lattice with skew quadrupolar field errors, what is the eigentune dependence at the order two in the perturbation? The resonance theory can immediately but partially answer to this problem (see e.g. [4]). Indeed, the results that it produces are only true in the neighbourhood of a sum or difference resonance $Q_x \pm Q_y = p$. Some residual effects are not predictable by this theory and simulations on LHC Version 6.-2 have shown that the latter could be non-negligible in terms of Q'' and in view of the LHC tolerance at injection $Q'' < 1000$ [1]. For this reason, the canonical perturbation method has been preferred. The basic principles of this method will be recalled in a first section, then applied to the problem of chromatic coupling. Results of simulations performed with MAD on the LHC lattice version 6.-2 will be presented and compared with the ones analytically obtained.

2.1 Reminders

In this section, we will partly follow Ref. [5].
Let us consider a $2d$-dimensional dynamical system described by the following Hamiltonian

$$H(\Phi, \mathbf{J}, s) = H_0(\mathbf{J}, s) + \epsilon\, V(\Phi, \mathbf{J}, s)\ , \tag{4}$$

where H has been written in terms of the action-angle variables $(\mathbf{J}, \Phi)$ (bold-face characters denote d-dimensional vectors) of the unperturbed motion defined by the Hamiltonian H_0. The following assumptions are made: the Hamiltonian H_0 as well as the perturbation V are periodic in s of period C; V is 2π periodic in the angle variable Φ and has zero average value with respect to it.

$$V(\Phi, \mathbf{J}, s) = \sum_{\mathbf{m}} v_{\mathbf{m}}(\mathbf{J}, s)\, \exp(i\,\mathbf{m}\cdot\Phi)\ \text{ with }\ v_0(\mathbf{J}, s) = \frac{1}{(2\pi)^d}\int_0^{2\pi} d\Phi\, V(\Phi, \mathbf{J}, s) = 0\,. \tag{5}$$

If V has nonzero average, the average value of V can always be absorbed into H_0 (this requirement will be justified later). Finally, we note $\boldsymbol{\nu}(\mathbf{J}, s)$ and $\mathbf{Q}(\mathbf{J})$ the local vector frequency and the eigentunes associated to the unperturbed Hamiltonian H_0:

$$\boldsymbol{\nu}(\mathbf{J}, s) \stackrel{\text{def}}{=} \partial_{\mathbf{J}} H_0(\mathbf{J}, s) \quad \text{and} \quad \mathbf{Q}(\mathbf{J}) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_0^C ds\, \boldsymbol{\nu}(\mathbf{J}, s)\ . \tag{6}$$

The aim of the method is to find a canonical transformation, $(\mathbf{J}, \Phi) \mapsto (\mathbf{J_1}, \Phi_{\mathbf{1}})$, which gives to the new Hamiltonian H_1 the following property: all the terms occurring in H_1 and induced by the perturbation are at least of order ϵ^2:

$$\begin{aligned} &H_1(\Phi_{\mathbf{1}}, \mathbf{J_1}, s) = H_0(\mathbf{J_1}, s) + \epsilon^2\, \Delta H_0(\mathbf{J_1}, s) + \epsilon^2\, V_1(\Phi_{\mathbf{1}}, \mathbf{J_1}, s, \epsilon) \\ &\text{with} \int_0^{2\pi} d\Phi_{\mathbf{1}}\ V_1(\Phi_{\mathbf{1}}, \mathbf{J_1}, s, \epsilon{=}0)\ = 0\ . \end{aligned} \tag{7}$$

Under these conditions, the new action $\mathbf{J_1}$ becomes an invariant of motion at the second order in the perturbation and the new eigentunes $\mathbf{Q_1}(\mathbf{J_1})$ can be computed up to the order ϵ^2:

$$\mathbf{Q_1}(\mathbf{J_1}) = \mathbf{Q}(\mathbf{J_1}) + \frac{\epsilon^2}{2\pi} \int_0^C ds\ \partial_{\mathbf{J_1}} \Delta H_0(\mathbf{J_1}, s) + \circ(\epsilon^2)\ . \tag{8}$$

The generating function of this transformation is searched as a function of the old coordinates Φ, the new momenta $\mathbf{J_1}$ and the independent variable s; it has to be close to the identity, 2π-periodic in Φ and C-periodic in s:

$$F_2(\Phi, \mathbf{J_1}, s) = \Phi \cdot \mathbf{J_1} + \epsilon\, G(\Phi, \mathbf{J_1}, s)\ , \tag{9}$$

$$\text{with } G(\Phi, \mathbf{J_1}, s) = \sum_{\mathbf{m}} g_{\mathbf{m}}(\mathbf{J_1}, s)\ \exp(i\, \mathbf{m} \cdot \Phi)\ . \tag{10}$$

The relations between old and new variables and the new Hamiltonian H_1 are then :

$$\begin{aligned} \Phi_{\mathbf{1}} &= \Phi + \epsilon\, G_{\mathbf{J_1}} \\ \mathbf{J} &= \mathbf{J_1} + \epsilon\, G_{\Phi} \\ H_1 &= H + \epsilon\, G_s \end{aligned} \tag{11}$$

where the subscripts represent partial differentiations. By continuing to use the mixed variables $(\mathbf{J_1}, \Phi)$, the new Hamiltonian H_1 can be expressed in the following form:

$$\begin{aligned} H_1 &= H_0\left(\mathbf{J_1} + \epsilon\, G_{\Phi}, s\right) + \epsilon\, V\left(\Phi, \mathbf{J_1} + \epsilon\, G_{\Phi}, s\right) + \epsilon\, G_s(\Phi, \mathbf{J_1}, s) \\ &= H_0(\mathbf{J_1}, s) + \epsilon\, \left[\boldsymbol{\nu}(\mathbf{J_1}, s) \cdot G_{\Phi}(\Phi, \mathbf{J_1}, s) + G_s(\Phi, \mathbf{J_1}, s) + V\left(\Phi, \mathbf{J_1}, s\right)\right] + \\ &\quad \epsilon^2 \underbrace{\left[\frac{1}{2}\, G_{\Phi}(\Phi, \mathbf{J_1}, s) \cdot \boldsymbol{\nu}_{\mathbf{J}}(\mathbf{J_1}, s) \cdot G_{\Phi}(\Phi, \mathbf{J_1}, s) + V_{\mathbf{J}}\left(\Phi, \mathbf{J_1}, s\right) \cdot G_{\Phi}(\Phi, \mathbf{J_1}, s) + \circ(1)\right]}_{=\left[\frac{1}{2}\, G_{\Phi} \cdot \boldsymbol{\nu}_{\mathbf{J}} \cdot G_{\Phi} + V_{\mathbf{J}} \cdot G_{\Phi}\right]_{\left(\Phi_{\mathbf{1}}, \mathbf{J_1}, s\right)} +\ (1)}. \end{aligned} \tag{12}$$

Consequently, if the function G verifies the equation

$$\boldsymbol{\nu}(\mathbf{J_1}, s) \cdot G_\Phi + G_s + V\,(\Phi, \mathbf{J_1}, s) = 0\ , \tag{13}$$

the new Hamiltonian H_1 takes the form of Eq. (7) with

$$\left\{\begin{array}{lcl} \Delta H_0(\mathbf{J_1}, s) & = & \dfrac{1}{(2\pi)^d}\displaystyle\int_0^{2\pi} d\Phi_\mathbf{1}\ \left[\frac{1}{2}\, G_\Phi \cdot \boldsymbol{\nu}_\mathbf{J} \cdot G_\Phi + V_\mathbf{J} \cdot G_\Phi\right]_{(\Phi_\mathbf{1}, \mathbf{J_1}, s)} \\ V_1(\Phi_\mathbf{1}, \mathbf{J_1}, s, \epsilon) & = & \left[\dfrac{1}{2}\, G_\Phi \cdot \boldsymbol{\nu}_\mathbf{J} \cdot G_\Phi + V_\mathbf{J} \cdot G_\Phi\right]_{\left(\Phi_\mathbf{1}, \mathbf{J_1}, s\right)} - \Delta H_0(\mathbf{J_1}, s) + \circ(1)\ . \end{array}\right. \tag{14}$$

By using the relations (5) and (10), the partial derivative equation (13) can be expressed in terms of the harmonics $g_\mathbf{m}$ and $v_\mathbf{m}$:

$$\left[i\,\mathbf{m} \cdot \boldsymbol{\nu}(\mathbf{J_1}, s) + \partial_s\right] g_\mathbf{m} = -v_\mathbf{m} \tag{15}$$

Outside the separatrices $\mathbf{m} \cdot \mathbf{Q}(\mathbf{J_1}) \equiv$ *integer*, periodic solutions always exist even for $\mathbf{m} = 0$ since, by assumption, the harmonic v_0 is zero (see Eq. 2); they are given by

$$g_\mathbf{m}(\mathbf{J_1}, s) = \frac{i/2}{\sin\left(\pi\,\mathbf{m} \cdot \mathbf{Q}(\mathbf{J_1})\right)} \int_s^{s+C} ds'\ \exp\Big(i\mathbf{m} \cdot \left(\Psi(\mathbf{J_1}, s') - \Psi(\mathbf{J_1}, s) - \pi\mathbf{Q}(\mathbf{J_1})\right)\Big)\, v_\mathbf{m}(\mathbf{J_1}, s') \tag{16}$$

$$\text{where } \Psi(\mathbf{J_1}, s) \stackrel{\text{def}}{=} \int_0^s ds'\ \boldsymbol{\nu}(\mathbf{J_1}, s')\ . \tag{17}$$

By using the relations (16) (giving G) and (14) (giving ΔH_0 as a function of G), the dependence of the eigentunes at the second order in the perturbation are obtained thanks to Eq. (8). In the particular case where H_0 varies linearly with J , the vectors $\boldsymbol{\nu}(\mathbf{J}, s)$, $\mathbf{Q}(\mathbf{J})$ and $\Psi(\mathbf{J}, s)$ becomes independent of the action variable; in that case, the second derivative with respect to ϵ of the perturbed eigentunes takes the following simplified form:

$$\left(\frac{\partial^2 \mathbf{Q_1}}{\partial \epsilon^2}\right)(\mathbf{J_1}) = \sum_\mathbf{m} \frac{-\quad \mathbf{J_1}}{\mathbf{m} \cdot \mathbf{Q}} \int_0^C \int_s^{s+C}{}' \left[\mathbf{m} \cdot \left(\frac{-\mathbf{m}}{\mathbf{J}}\right)_{(\mathbf{J_1}, s)} {}_\mathbf{m}\ \mathbf{J_1}\quad {}'\qquad \Big(\mathbf{m} \cdot \quad {}' - \quad - \mathbf{Q}\Big)\right]. \tag{18}$$

2.2 Application to chromatic coupling and link with the resonance theory

This method is particularly well-suited to study the dynamics of a particle of energy $p_0(1+\delta)$ travelling through a ring which presents skew sextupolar field errors.
Firstly, let us begin with some recalls concerning the unperturbed linear problem. If the purely chromatic focusing effects are neglected (being corrected by the lattice sextupoles), the Hamiltonian H_0 associated to the unperturbed dynamics of this particle and written in terms of action-angle variables $(J_{x,y}, \phi_{x,y})$ is given by the usual expression (see e.g. [5])

$$H_0(J_x, J_y, s) = \frac{J_x}{\beta_x(s)} + \frac{J_y}{\beta_y(s)}\ , \tag{19}$$

where $\beta_{x,y}$ are the horizontal and vertical β functions. The local vector frequency $\boldsymbol{\nu}(\mathbf{J},s) \stackrel{\text{def}}{=} \partial_{\mathbf{J}} H_0(\mathbf{J},s) = \Big(1/\beta_x(s)\,,\ 1/\beta_y(s)\Big)$ is then independent of the action variables and the functions $\Psi_{x,y}(s)$ introduced in Eq. (17) describe the betatron phase advances in the horizontal and vertical planes:

$$\Psi_{x,y}(s) \equiv \mu_{x,y}(s) = \int_0^s \frac{ds'}{\beta_{x,y}(s')} \ . \tag{20}$$

Let us study now the effects generated by the distribution $K_2\ (s)$ of skew sextupolar field errors. By neglecting the geometric aberrations induced, this perturbation is equivalent to the one produced by a distribution $K_1\ (s)$ of skew quadrupoles where the strengths $K_1\ (s)$ $[\mathrm{m}\ ^{2}]$ and $K_2\ (s)$ $[\mathrm{m}\ ^{3}]$ are linked by Eq. (3). The potential which adds up to the unperturbed Hamiltonian is then

$$\delta * V = \delta * D_x(s)\, K_2\ (s)\, xy = \delta * 2\sqrt{\beta_x(s)\beta_y(s)}\,\sqrt{J_x J_y}\, K_2\ (s) D_x(s)\, \cos(\phi_x)\cos(\phi_y)\ , \tag{21}$$

where the relations linking spatial coordinates (x,y) and action-angle variables $(J_{x,y},\phi_{x,y})$ have been used[1].
Therefore, the momentum deviation δ plays here the role of the parameter ϵ and the harmonics $v_{\mathbf{m}}$ associated to the potential V are all zero except for $\mathbf{m} = (\pm 1, \pm 1)$:

$$v_{\ 1,\ 1}(J_x, J_y, s) = \frac{1}{2}\sqrt{\beta_x(s)\beta_y(s)}\,\sqrt{J_x J_y}\, K_2\ (s) D_x(s)\ . \tag{22}$$

By using Eq. (18) and after some trigonometric manipulations, the second order chromaticities induced by the skew sextupole distribution $K_2\ (s)$ are easily obtained:

$$\begin{aligned} Q''_{I,II} \ &= \ \pm \frac{1}{8\pi \sin\Big[\pi(Q_x - Q_y)\Big]} \int_0^C ds \int_s^{s+C} ds'\, f(s) f(s') \cos\big(\mu\ \ (s,s')\big) - \\ &\qquad \frac{1}{8\pi \sin\Big[\pi(Q_x + Q_y)\Big]} \int_0^C ds \int_s^{s+C} ds'\, f(s) f(s') \cos\big(\mu^{+}(s,s')\big) \end{aligned} \tag{23}$$

$$\text{where} \left\{ \begin{array}{ccl} \mu\ \ (s,s') & \stackrel{\text{def}}{=} & (\mu_x(s') - \mu_x(s) - \pi Q_x) \pm (\mu_y(s') - \mu_y(s) - \pi Q_y) \\ f(s) & \stackrel{\text{def}}{=} & \sqrt{\beta_x(s)\beta_y(s)}\, D_x(s)\, K_2\ (s)\ . \end{array} \right. \tag{24}$$

Note that the eigentunes of the new normal modes I and II must be independent of the action variable since the perturbation considered here is of quadrupolar type. This is checked by the previous relations at the second order in the perturbation.
By using the periodicity of the optical functions $\beta_{x,y}$ and D_x and the fact that $\mu_{x,y}(s+C) = \mu_{x,y}(s) + 2\pi Q_{x,y}$, the previous expressions can be split into two distinct parts, the first

[1] $\quad = \sqrt{\quad_{x,y}\quad_{x,y}}\qquad_{x,y}$

one containing resonant terms (i.e. infinitely growing in the neighbourhood of a sum or difference resonance $Q_x \pm Q_y = p$) and the other being bounded independently of the tunes:

$$Q''_{I,II} = \frac{\pi}{2}\left[\pm\cot\Big[\pi(Q_x - Q_y)\Big]\,|c\ \ |^2 - \cot\Big[\pi(Q_x + Q_y)\Big]\,|c_+|^2\right] - \pi\left[\pm d\ \ - d_+\right] \qquad (25)$$

$$\text{with}\quad \begin{cases} c & \stackrel{\text{def}}{=} \dfrac{1}{2\pi}\displaystyle\int_0^C ds\, f(s)\, e^{i(\mu_x(s)\ \ \mu_y(s))} \\ d & \stackrel{\text{def}}{=} \dfrac{1}{4\pi^2}\,\Im\left[\displaystyle\int_0^C ds\, f(s)\, e^{\ \ i(\mu_x(s)\ \ \mu_y(s))}\int_0^s ds'\, f(s')\, e^{i(\mu_x(s')\ \ \mu_y(s'))}\right], \end{cases} \qquad (26)$$

where the symbol $\Im$ indicates the imaginary part.
Let us compare this results with the ones expected by the resonance theory, when, for instance, the tunes Q_x and Q_y are localised in the neighbourhood of a difference resonance $Q_x - Q_y = p$ with $\Delta \stackrel{\text{def}}{=} Q_x - Q_y - p \ll 1$. By using $\cot\Big[\pi(Q_x - Q_y)\Big] \sim (\pi\Delta)^{\ \ 1}$ in Eq. (25), the canonical perturbation method yields

$$Q''_{I,II} \sim \pm\frac{|c\ \ |^2}{2\Delta}\,, \qquad (27)$$

to be compared with the results announced by the resonance theory (see e.g. [4]):

$$Q_{I,II} = \frac{1}{2}\left(Q_x + Q_y - p\right) \pm \frac{1}{2}\,\frac{\Delta}{|\Delta|}\sqrt{\Delta^2 + |\kappa|^2}\ \ \text{mod}[1]\,, \qquad (28)$$

where the complex coefficient κ is given by $\kappa \stackrel{\text{def}}{=} \dfrac{1}{2\pi}\displaystyle\int_0^C ds\,\sqrt{\beta_x\beta_y}\,K_1\ \ e^{i(\mu_x\ \ \mu_y\ \ 2\pi\Delta s/C)} \sim c\ \ \delta$.
The second order chromaticities obtained by this way are then identical to the ones given by Eq. (27):

$$Q_{I,II} \sim \frac{1}{2}\left(Q_x + Q_y - p\right) \pm \frac{\Delta}{2} \pm \frac{|c\ \ |^2}{4\Delta}\,\delta^2\ \ \text{mod}[1]\ \ \text{leading to}\ \ Q''_{I,II} \sim \pm\frac{|c\ \ |^2}{2\Delta}\,. \qquad (29)$$

Nevertheless, on resonance, the canonical perturbation method is obviously inefficient since it produces diverging results for $Q''_{I,II}$. In that case, the resonance theory clarifies the problem: the eigentunes $Q_{I,II}(\delta)$ vary as $|\delta|$ (see Eq. (28) for $\Delta = 0$) and then are not differentiable for $\delta = 0$. On the other hand, in a real machine where the tunes do never coincide *exactly* with a difference resonance ($\Delta = -0.03$ for the LHC at injection), the canonical perturbation method gives more precise results as shown in Eq. (25) by the presence of the residual terms d .

2.3 Analytical calculation of the second order chromaticities induced by chromatic coupling in the LHC

The aim of this section is to use the previous results in order to obtain the analytical expressions of the second order chromaticities induced by chromatic coupling in the LHC. The terms c and d will be separately treated (Subsections 2.3.3 & 2.3.4). Beforehand, some preliminary simplifying assumptions concerning the LHC optics will be needed and the survey of the possible different sources of component a_3 in the ring has to be done.

2.3.1 LHC optics model, symmetries and notations

As said before, the chromatic coupling phenomena arise only in the dispersive regions of the arcs. Therefore, the skew sextupolar field errors will be neglected in the dispersion suppressors as well as in the eight experimental or service insertions. In other words, we will assume that the LHC is made of eight arc, each arc containing $N_{cell} = 23$ identical FODO cells, the insertions (included the dispersion suppressors) acting only as "phase trombones" with respect to chromatic coupling. Detailed calculations with MAD will demonstrate the validity of this approximation.
Starting from IP_1, the horizontal and vertical phase advances at mid-arc will be denoted by μ_{x_k} and μ_{y_k}, $k = 1 \cdots 8$. For reasons explained hereafter, we can reasonably assume that the LHC possesses a super-periodicity of 8 in $\mu_x - \mu_y$, that is

$$[\mu_{x_l} - \mu_{y_l}] - [\mu_{x_k} - \mu_{y_k}] = \frac{2\pi\, p_0}{8}\ (l-k)\ \ , 1 \leq k \leq l \leq 8\,, \tag{30}$$

where $p_0 \equiv Q_x - Q_y$ is the tune split ($p_0 = 4$ or 5 for LHC Version 6.-2). Indeed, on the one hand, in each of the eight insertions (excluded the dispersion suppressors), and because of the antisymmetric design of their optics, the horizontal and the vertical phase advances are bound to be identical. On the order hand, in each octant, the main role of the dispersion suppressors is exactly the same: they suppress the horizontal dispersion generated in the arc by over-focusing in the horizontal plane. Therefore, the natural tune split Δp_0 induced by the DS's is roughly the same in each octant. In LHC Version 6.-2, this quantity is approximatively equal to 0.25 per octant.
Finally, by using this last remark as well as Eq. (30), the horizontal and vertical phase advances per cell, $\boldsymbol{\mu_x}$ and $\boldsymbol{\mu_y}$, can be linked by the following relation:

$$N_{cell}\,(\boldsymbol{\mu_x} - \boldsymbol{\mu_y}) + 2\pi\,\Delta p_0 \sim \frac{2\pi\, p_0}{8} \Rightarrow N_{cell}\,\frac{\boldsymbol{\mu_x} - \boldsymbol{\mu_y}}{2} \sim \pi\,\frac{p_0 - 2}{8}\ , \tag{31}$$

assuming that the value $\Delta p_0 \sim 0.25$ remains quasi-constant for a reasonable choice of the tune split p_0, which seems to be the case in the LHC.

2.3.2 Various sources of a_3 in the LHC

Let us continue the discussion by making the survey of the different sources of a_3 existing or liable to be induced in the arc magnets by feed-down effect. In terms of integrated strengths, we will show that, regardless of the energy, the component a_3 of geometric origin in the main dipoles is indisputably the most critical.
Concerning the number and the gradients of the different magnets which will be considered, we will refer to the injection and collision optics of the LHC version 6.-2 (see Tab. 1). The lattice octupoles will be assumed to be switched off at injection and of maximum strength at the collision energy.
We will use the error table **9712** for the field imperfections expected at injection and collision in the main dipoles MB and main quadrupoles MQ (see Tab. 2).
The components a_3 induced by closed orbit deviations (e.g. by vertical orbit deviations in lattice octupoles) or magnet misalignments (e.g. by rolls of lattice sextupoles) will be also included. The alignment tolerances considered here are reported in Tab. 3. They are

all based on private communications [6] in absence of an official error table. Unlike the specifications concerning the field imperfections, the misalignments will be described by only two components: a random component and a systematic component per arc. Finally, the horizontal and vertical closed orbit deviations, assumed to be purely random, will be taken equal to 1 mm r.m.s. in the arcs.
The results obtained are summarised in Tab. 4 and expressed in terms of equivalent integrated skew sextupolar strength $\left(K_2\ L\right)_{eq}$, both for the injection and collision energies. Without detailing the calculations, we give hereafter the different sources of chromatic coupling which have been considered:

- for the main dipoles, the imperfection a_3, the imperfections b_3, a_4 and b_4 combined with the dipole misalignments as well as with orbit deviations in the dipole.

- for the b_3 and b_4 spool pieces, the misalignments (roll angle and vertical displacement respectively) with respect to the dipole.

- for the main quadrupoles, the imperfection a_3, the imperfections b_3, a_4 and b_4 combined with the quadrupole misalignments as well as with closed orbit deviations in the quadrupole.

- the roll of the chromatic correction sextupoles.

- the vertical misalignments combined with the vertical orbit deviations in the arc octupoles.

On the other hand, we have to specify the computation rules which have been adopted:

- the different sources of systematic a_3, induced by systematic multi-polar components (i.e mean+uncertainty) combined with a systematic displacement have been added linearly (which is pessimistic).

- the different sources of random a_3, induced by systematic (resp. random) multi-polar components combined with a random (resp. systematic or random) misalignment have been added quadratically.

As shown in Tab. 4 and in terms of integrated skew sextupolar strength per arc, the contribution of the lattice sextupoles and quadrupoles is irrelevant and will be neglected (100 times lower than the dipole one). The systematic contribution of the octupoles (when they are switched on at collision, pushed to their maximum strength and systematically misaligned by 0.5 mm vertically) represents 6.2 % of the total skew sextupolar strength integrated per arc at collision. Insofar as the second order chromaticity induced scales quadratically with a_3, this contribution will be also neglected. In conclusion and according to this study, the only skew sextupolar error sources that will consider in the rest of this paper will be the ones induced in the main dipoles at injection (the values relative to the collision energy being slightly lower): the systematic component $(K_2\ L)^{(sys)} = 3.34\,10^{\ 3}$ m$^{\ 2}$ (corresponding to $a_3 = 0.946$) and the random component $(K_2\ L)^{(ran)} = 1.85\,10^{\ 3}$ m$^{\ 2}$ (corresponding to $a_3 = 0.524$) which, due to feed-down effects, have been increased by less than 10 %.

Type of magnet	$(K_iL)[m^{\ i}]$ at Injection	$(K_iL)[\mathrm{m}^{\ i}]$ at Collision	Magnetic length [m]	Number per arc (excluded the DS's)
Main dipoles MB	$5.1\,10^{\ 3}$	$5.1\,10^{\ 3}$	14.3	138
Correctors b_3 MCS	$\times$	$\times$	0.1	138
Correctors $b_4 + b_5$ MCDO	$\times$	$\times$	0.063	138
Main quads QF	0.027	0.027	3.1	23 or 24
Main quads QD	-0.026	-0.026	3.1	23 or 24
Arc sextupoles MSCH	0.019	0.032	1.1	23 or 24
Arc sextupoles MSCV	-0.037	-0.061	1.1	23 or 24
Arc octupoles MO	0	± 5.51	0.32	11

Table 1: The main magnets of the arcs : number per arc, magnetic length and integrated strength at injection and collision in LHC Version 6.-2

$\mathbf{a_n}$ & $\mathbf{b_n}$	Main dipole MB Sum of Persistent & Geometric			Main quadrupole MQ Sum of Persistent & Geometric		
	Mean (outer/ inner channel)	Uncertainty (max. value)	Random (r.m.s)	Mean (outer/ inner channel)	Uncertainty (max. value)	Random (r.m.s)
At Injection						
a_3	0.000/0.000	± 0.867	0.478	0.000/0.000	± 0.867	1.445
b_3	$-7.774/-7.774$	± 1.377	1.474	0.000/0.000	± 0.867	1.445
a_4	0.000/0.000	± 0.491	0.513	0.000/0.000	± 0.983	0.737
b_4	$0.133/-0.133$	± 0.344	0.491	0.000/0.000	± 0.983	0.737
At Collision						
a_3	0.000/0.000	± 0.867	0.433	0.000/0.000	± 0.867	1.445
b_3	2.890/2.890	± 0.867	1.445	0.000/0.000	± 0.867	1.445
a_4	0.000/0.000	± 0.491	0.491	0.000/0.000	± 0.983	0.737
b_4	$0.133/-0.133$	± 0.344	0.491	0.000/0.000	± 0.983	0.737

Table 2: Expected sextupolar and octupolar components in the MB's and MQ's at injection and collision (error table **9712** in units of $10^{\ 4}$ relative field error at a radius of 17 mm)

Type of magnet	Systematic per arc (max. value)			Random (r.m.s)		
	Δx [mm]	Δy [mm]	$\Delta\phi$ [mrad]	Δx [mm]	Δy [mm]	$\Delta\phi$ [mrad]
MB	± 0.5	± 0.5	± 1.0	0.5	0.5	1.0
MCS w.r.t. MB	$\times$	$\times$	± 1.0	$\times$	$\times$	1.0
MCDO w.r.t. MB	$\times$	± 0.3	$\times$	$\times$	0.5	$\times$
QF/QD	± 0.5	± 0.5	± 1.0	0.5	0.5	1.0
MSCH/MSCV	$\times$	$\times$	± 1.0	$\times$	$\times$	1.0
MO	$\times$	± 0.5	$\times$	$\times$	0.5	$\times$

Table 3: Wildly estimated alignment tolerances of the arc magnets (the symbol $\times$ indicates that the misalignment considered does not induce skew sextupolar field error)

Type of magnet	Systematic per arc (max. value)	Random (r.m.s)
$\left(K_2\ L\right)_{eq}$ [m ^{2}] at injection		
MB + MCS + MCDO	$\pm 3.34\,10^{3}$	$1.85\,10^{3}$
QF/QD	$\pm 1.95\,10^{4}$ / $\pm 1.87\,10^{4}$	$2.78\,10^{4}$ / $2.68\,10^{4}$
MSCH/MSCV	$\pm 1.90\,10^{5}$ / $\pm 3.70\,10^{5}$	$1.90\,10^{5}$ / $3.70\,10^{5}$
MO	0.00	0.00
$\left(K_2\ L\right)_{eq}$ [m ^{2}] at collision		
MB + MCS + MCDO	$\pm 3.32\,10^{3}$	$1.66\,10^{3}$
QF/QD	$\pm 1.95\,10^{4}$ / $\pm 1.87\,10^{4}$	$2.78\,10^{4}$ / $2.68\,10^{4}$
MSCH/MSCV	$\pm 3.20\,10^{5}$ / $\pm 6.10\,10^{5}$	$3.20\,10^{5}$ / $6.10\,10^{5}$
MO	$\pm 2.75\,10^{3}$	$2.75\,10^{3}$

Table 4: Skew sextupole field errors liable to be induced in the arc magnets at injection and collision (a_3 + feed-down due to b_3, b_4 and a_4)

2.3.3 Calculation of the sum and difference coupling coefficients c

Let us now estimate the respective contributions of the systematic and random components a_3 of the dipoles to the second order chromaticity induced in the LHC. For this, we begin by evaluating the complex coefficients c given in Eq. (26). Given the properties of the unperturbed phase advances in the LHC (subsection 2.3.1), it is natural to split the ring integral defining these coefficients into 8 integrals over the 23 cells of each arc:

$$c \quad = \sum_{k=1}^{8} e^{i(\mu_{x_k} \quad \mu_{y_k})}\, c^{(k)} \;\text{ with }\; c^{(k)} \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{k^{th}\,arc} ds\, \sqrt{\beta_x \beta_y}\, \Big(\big(K_2\,\big)_k^{(sys)} + \big(K_2\,\big)_k^{(ran)} \Big) D_x\, e^{i(\mu_x \quad \mu_y)}\,, \tag{32}$$

where $\big(K_2\,\big)_k^{(sys)}$ (resp. $\big(K_2\,\big)_k^{(ran)}$) represents the systematic (resp. random) distribution of skew sextupolar field in the arc number k. In the definition of the coefficient $c^{(k)}$ relative to the k^{th} arc, note that the origin of the phases μ_x and μ_y is taken at the middle of the arc considered.

Contribution of the systematic component a_3 of dipoles

In a first time, we only consider the systematic component a_3 of dipoles in the calculation of the complex coefficients $c^{(k)}$. The latter can be obtained from the contribution $\Delta c^{(k)}$ of the central arc cell:

$$\Delta c^{(k)} \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{\quad L}^{L} ds\, \sqrt{\beta_x \beta_y}\; \big(K_2\,\big)_k^{(sys)}\; D_x\, e^{i(\mu_x \quad \mu_y)} \tag{33}$$

(where L is the half-cell length), and

$$c^{(k)} = \sum_{m=\quad(N_{cell}\quad 1)/2}^{(N_{cell}\quad 1)/2} e^{i\,m\,\mu_\pm}\, \Delta c^{(k)} = f \quad \Delta c^{(k)}\,, \tag{34}$$

where $\mu \quad \stackrel{\text{def}}{=} \boldsymbol{\mu_x} \pm \boldsymbol{\mu_y}$ denotes the sum or difference of the horizontal and vertical phase advances per cell and where

$$f \quad \stackrel{\text{def}}{=} \frac{\sin(N_{cell}\,\mu \quad /2)}{\sin(\mu \quad /2)}\,. \tag{35}$$

By noting that the phases $\boldsymbol{\mu_x}$ and $\boldsymbol{\mu_y}$ are approximatively equal to 90 for the LHC and that the number of cells is odd, the arc integral defining the coefficient $c_+^{(k)}$ is about equal to the contribution of the central cell, i.e.

$$f_+ \sim 1\,. \tag{36}$$

On the other hand, by using Eq. (31) linking the phase advance difference per cell μ to the tune split p_0, we obtain directly

$$f \quad \sim \frac{\sin\left(\dfrac{p_0 - 2}{8}\,\pi\right)}{\sin\left(\dfrac{p_0 - 2}{8\,N_{cell}}\,\pi\right)}\,. \tag{37}$$

Consequently, for a nominal tune split of 4 or 5, the coefficient f is roughly equal to the number of cells per arc, i.e. $N_{cell} = 23$; on the other hand, f is vanishing when the phase advance difference per arc, i.e. $N_{cell}(\boldsymbol{\mu_x} - \boldsymbol{\mu_y})$, is a integer multiple of 2π, that is, for instance, for a tune split of 10 (see Ref. [7] for a more general proof of this result valid in the frame of the single resonance theory).
The contribution of the central arc cell is shown in Appendix A and reported hereafter. For a sake of simplification, this computation has been carried out in the thin lens approximation; in each half-cell of length L, the three dipoles have been replaced by an equivalent single dipole, the bending angle of which is $\alpha = 6\pi/1232$; finally, the phase advances in both transverses planes have been assumed to be rigorously the same ($K_F \equiv -K_D$), say equal to $\mu = (\boldsymbol{\mu_x} + \boldsymbol{\mu_y})/2$. Due to the symmetry according to the central quadrupole of the cell, the coefficients $\Delta c^{(k)}$ are real and given by

$$
\begin{aligned}
\Delta c^{(k)} &= \frac{1}{4\pi}\,\frac{\alpha\,L^2}{\sin^3(\mu/2)}\Big(12-\sin^2(\mu/2)\Big)\ \big(K_2\ L\big)_k^{(sys)} \\
\Delta c_+^{(k)} &= \begin{cases} \dfrac{1}{4\pi}\,\dfrac{\alpha\,L^2}{\sin^3(\mu/2)}\Big(12-9\sin^2(\mu/2)+2\sin^3(\mu/2)+3/5\ \sin^4(\mu/2)\Big)\ \big(K_2\ L\big)_k^{(sys)} \\ \text{if the central quadrupole of the } k^{th} \text{ arc is focusing,} \\ \dfrac{1}{4\pi}\,\dfrac{\alpha\,L^2}{\sin^3(\mu/2)}\Big(12-9\sin^2(\mu/2)-2\sin^3(\mu/2)+3/5\ \sin^4(\mu/2)\Big)\ \big(K_2\ L\big)_k^{(sys)} \\ \text{if the central quadrupole of the } k^{th} \text{ arc is defocusing,} \end{cases}
\end{aligned}
\tag{38}
$$

where $\big(K_2\ L\big)_k^{(sys)}$ represents the integrated skew sextupolar strength per dipole, systematic in the arc number k. Taking $\mu \sim \pi/2$ and $L = 53.45$ m, the previous relations yield

$$
\begin{aligned}
\Delta c^{(k)} &\sim 113.14 \times \big(K_2\ L\big)_k^{(sys)}\ [\mathrm{m}\ \ ^2] \\
\Delta c_+^{(k)} &\sim \begin{cases} 82.22 \times \big(K_2\ L\big)_k^{(sys)}\ [\mathrm{m}\ \ ^2] \\ 68.31 \times \big(K_2\ L\big)_k^{(sys)}\ [\mathrm{m}\ \ ^2]\,. \end{cases}
\end{aligned}
\tag{39}
$$

Therefore, for tune splits of a few units, say 4 or 5 (which correspond to the nominal values of the LHC versions 5 and 6), the ratio $c_+^{(k)}/c^{(k)}$ scales roughly as $f_+/f \sim 1/N_{cell}$. In terms of second order chromaticity (see Eq. (25)), this is equivalent to a difference of four orders of magnitude between these two contributions, knowing that the fractional parts of the horizontal and vertical tunes are .28 and .31 respectively for the nominal injection optics of the LHC.
Concerning the effects induced by the systematic component a_3 of dipoles, the sum coupling coefficient c_+ will then be neglected. On the other hand, by using the results obtained in this subsection as well as Eq. (30), the difference coupling coefficient c is given by the

following expression

$$|c\;\;| \sim \frac{\alpha\, L^2}{4\pi\, \sin^3(\mu/2)} \Big(12 - \sin^2(\mu/2)\Big) \; \frac{\sin\left(\dfrac{p_0-2}{8}\,\pi\right)}{\sin\left(\dfrac{p_0-2}{8\,N_{cell}}\,\pi\right)} \left|\sum_{k=1}^{8} e^{i\frac{2\pi p_0 k}{8}} \,\big(K_2\; L\big)_k^{(sys)}\right| , \qquad (40)$$

with $N_{cell} = 23$, $L = 53.45$ m, $\alpha = 6\pi/1232$ and $\mu \sim \pi/2$.
For a given tune split p_0, we show hereafter that it exists at least one specific set of values $\big((K_2\; L)_k^{(sys)}\big)_{(1\;\; k\;\; 8)}$ which maximises the coefficient $|c\;\;|$ (worst case). For this, we write

$$\big(K_2\; L\big)_k^{(sys)} = \sum_{p=0}^{7} \lambda_p \, e^{\;\; 2i\pi pk/8} \text{ with } \lambda_p = \frac{1}{8} \sum_{k=1}^{8} \big(K_2\; L\big)_k^{(sys)} e^{2i\pi pk/8} . \qquad (41)$$

Since the set $\big((K_2\; L)_k^{(sys)}\big)_{(1\;\; k\;\; 8)}$ is real, the harmonics λ_p satisfy

$$\lambda_{0,4} = \overline{\lambda}_{0,4} \text{ and } \lambda_p = \overline{\lambda}_{8\;\; p} \text{ for } p \neq 0 \text{ and } p \neq 4 . \qquad (42)$$

Thus, given the tolerance on the systematic component a_3 (expressed in terms of integrated strength $\big(K_2\; L\big)_{max}^{(sys)}$, maximum per dipole), an upper bound on the harmonics λ_p can be obtained by using the Parceval theorem:

$$\sum_{p=0}^{7} |\lambda_p|^2 = |\lambda_0|^2 + |\lambda_4|^2 + 2\sum_{p=1}^{3} |\lambda_p|^2 = \frac{1}{8}\sum_{k=1}^{8} \left|\big(K_2\; L\big)_k^{(sys)}\right|^2 \leq \left[\big(K_2\; L\big)_{max}^{(sys)}\right]^2 , \qquad (43)$$

leading to $|\lambda_{0,4}| \leq \big(K_2\; L\big)_{max}^{(sys)}$ and $|\lambda_p| \leq 1/2\; \big(K_2\; L\big)_{max}^{(sys)}$ for $p \neq 0$ and $p \neq 4$.
By coming back to Eq. (40), we obtain finally the following results:

- if $p_0 \equiv 0\;\;[8]$, $\left|\sum_{k=1}^{8} e^{i\frac{2\pi p_0 k}{8}} \,\big(K_2\; L\big)_k^{(sys)}\right| = 8|\lambda_0| \leq \mathbf{8}\,\big(K_2\; L\big)_{max}^{(sys)}$ and the worst case is obtained when the systematic component a_3 is the same in each octant:

$$\big(K_2\; L\big)_k^{(sys)} = \pm\,\big(K_2\; L\big)_{max}^{(sys)} \;\; , \;\; k = 1 \cdots 8 \; .$$

- if $p_0 \equiv 4\;\;[8]$, $\left|\sum_{k=1}^{8} e^{i\frac{2\pi p_0 k}{8}} \,\big(K_2\; L\big)_k^{(sys)}\right| = 8|\lambda_4| \leq \mathbf{8}\,\big(K_2\; L\big)_{max}^{(sys)}$ and the worst case is obtained when the systematic component a_3 is alternated from one arc to the other:

$$\big(K_2\; L\big)_k^{(sys)} = \pm(-1)^k\,\big(K_2\; L\big)_{max}^{(sys)} \;\; , \;\; k = 1 \cdots 8 \; .$$

- if $p_0 \equiv \pm p$ [8] with $p \neq 0, 4$, $\left|\sum_{k=1}^{8} e^{i\frac{2\pi p_0 k}{8}} \,\big(K_2\; L\big)_k^{(sys)}\right| = 8|\lambda_p| = 8|\lambda_{8\;\; p}| \leq \mathbf{4}\,\big(K_2\; L\big)_{max}^{(sys)}$ and the worst case is obtained when only the harmonics λ_p and $\lambda_{8\;\; p}$ are nonzero:

$$\big(K_2\; L\big)_k^{(sys)} = \cos(2\pi p_0 k/8 + \varphi)\;\; \big(K_2\; L\big)_{max}^{(sys)} \; , \; k = 1 \cdots 8 \, .$$

For $p_0 = 4$ and $p_0 = 5$, the previous results yield (see Tab. 4)

$$\begin{cases} |c_{\ }|_{max} &= 18746.3 \times \left(K_2\, L\right)^{(sys)}_{max} \ [\mathrm{m}\ \ ^{2}] &= 62.61 \ \text{ for } \ p_0 = 4 \\ |c_{\ }|_{max} &= 8166.4 \times \left(K_2\, L\right)^{(sys)}_{max} \ [\mathrm{m}\ \ ^{2}] &= 27.28 \ \text{ for } \ p_0 = 5\ . \end{cases} \tag{44}$$

For these two cases, the contribution of the systematic component a_3 is then spectacular in terms of Q'' (see Eq. 25 with $Q_x - Q_y = -.03$ mod[1]):

$$\begin{cases} Q''_{I,II} &= \mp 65146 \quad \text{for} \ \ p_0 = 4 \\ Q''_{I,II} &= \mp 12363 \quad \text{for} \ \ p_0 = 5\ . \end{cases} \tag{45}$$

Note finally that the coefficient $c_{\ }$ is always vanishing in the case where the distribution $\left((K_2\, L)^{(sys)}_k\right)_{(1\ \ k\ \ 8)}$ does not contain the harmonic λ_p (as well as its conjugate $\lambda_{8\ \ p}$), where the integer p (in between 0 and 7) satisfy $p_0 \pm p \equiv 0\ \ [8]$.

Contribution of the random component a_3 of dipoles

Let us now estimate the standard deviations of the the sum and difference coupling coefficients $c_{\ }$ induced by the random component a_3 of dipoles. For this purpose, we write

$$c_{\ } = \frac{1}{2\pi} \int_0^C ds\, \sqrt{\beta_x \beta_y}\, D_x\, K_2\ \ e^{i(\mu_x\ \ \mu_y)} \sim \frac{1}{2\pi} \sum_{n\ \ dipoles} \left(K_2\, L\right)_n \sqrt{\beta_{x_n}\beta_{y_n}}\, D_{x_n}\, e^{i(\mu_{x_n}\ \ \mu_{y_n})}\ ,$$

where the optical functions β_{x_n,y_n} and D_{x_n} as well as the phase advances μ_{x_n,y_n} are taken at the centre of the dipole number n. Consequently, if $\left(K_2\, L\right)^{(rms)}$ represents the standard deviation of the integrated skew sextupolar strength per dipole (see Tab. 4), the previous equation yields

$$\begin{aligned} |c^{(rms)}_{\ }| &\stackrel{\text{def}}{=} \sqrt{\langle |c_{\ }|^2\rangle - |\langle c_{\ }\rangle|^2} = \frac{1}{2\pi}\, \left(K_2\, L\right)^{(rms)} \times \sqrt{\sum_{n\ \ dipoles} \beta_{x_n}\beta_{y_n}\, D^2_{x_n}} \\ &= \frac{1}{2\pi}\, \left(K_2\, L\right)^{(rms)} \times \sqrt{N_{arc}\, N_{cell}\, \mathcal{I}}\,, \end{aligned} \tag{46}$$

where the coefficient $\mathcal{I}$ corresponds to the contribution of a single arc cell (containing six dipoles):

$$\mathcal{I} \stackrel{\text{def}}{=} \sum_{n=1}^{6} \beta_{x_n}\beta_{y_n}\, D^2_{x_n} \sim \frac{6}{2L} \int_0^{2L} ds\ \beta_x(s)\beta_y(s)\, D^2_x(s)\ . \tag{47}$$

By using a simplified model to describe the LHC arc cell (see the previous paragraph or Appendix A), this integral has been computed with MAPLE [8], giving

$$\mathcal{I} = \frac{\alpha^2\, L^4}{420}\, \frac{280 + 1823 \cos^2\left(\frac{\mu}{2}\right) + 348 \cos^4\left(\frac{\mu}{2}\right) + 65 \cos^6\left(\frac{\mu}{2}\right) + 4 \cos^8\left(\frac{\mu}{2}\right)}{\cos^2\left(\frac{\mu}{2}\right) \sin^6\left(\frac{\mu}{2}\right)}\ . \tag{48}$$

For $\mu \sim \pi/2$, $L = 53.45$ m and $\alpha = 6\pi/1232$, and according to Tab. 4, we finally obtain

$$|c^{(rms)}_{\ }| = |c^{(rms)}_{+}| \sim 660.7 \times \left(K_2\, L\right)^{(rms)} \ [\mathrm{m}\ \ ^{2}] = 1.22\ , \tag{49}$$

which is negligible in terms of Q'' (compare with Eq. (44)).

2.3.4 Estimates of the residual terms d

We still have to calculate the non-resonant residual terms d in order to complete the estimate of the second order chromaticities induced by a_3 in the LHC.
The definition of these coefficients is recalled hereafter (see Eq. (26)):

$$d \;\stackrel{\text{def}}{=}\; \frac{1}{4\pi^2}\,\Im\left[\int_0^C ds\, f(s)\, e^{\;i(\mu_x(s)\quad \mu_y(s))}\int_0^s ds'\, f(s')\, e^{i(\mu_x(s')\quad \mu_y(s'))}\right]$$

$$\text{with } f(s) \stackrel{\text{def}}{=} \sqrt{\beta_x(s)\beta_y(s)}\, D_x(s)\, K_2\ (s)\ .$$

By replacing this double integral by a discrete sum over all the dipoles of the ring, it can be easily checked that, in this form, the coefficients d only contain cross-products involving the strengths of two *different* dipoles. The latter are then vanishing when averaged over the random distribution $(K_2\ )^{(ran)}$. This result can be directly transposed in terms of Q'', keeping in mind that the second order chromaticities depends linearly on these coefficients (see Eq. (25)). Finally, it is clear that the standard deviations $d^{(rms)}$ are at the very most of the order of $|c^{(rms)}|^2$, then negligible in terms of Q''. Therefore, we will restrict our study to the effects induced by the distribution $(K_2\ )^{(sys)}$ (systematic per arc).
By using the notations of Subsection 2.3.3, the coefficients d can be decomposed in the following way:

$$d \;= d^{(1)} + d^{(2)} + d^{(3)}$$

$$\text{with}\begin{cases} d^{(1)} \stackrel{\text{def}}{=} \displaystyle\sum_{1\quad k<l\quad 8} \sin\big((\mu_{x_k} \pm \mu_{y_k}) - (\mu_{x_l} \pm \mu_{y_l})\big) c^{(k)}\, c^{(l)} \\[2ex] d^{(2)} \stackrel{\text{def}}{=} \displaystyle\sum_{k=1}^{8} \left(\Delta c^{(k)}\right)^2 \times \sum_{1\quad n<m\quad N_{cell}} \sin\big((n-m)\,\mu\ \ \big) \\[2ex] d^{(3)} \stackrel{\text{def}}{=} \displaystyle\frac{N_{cell}}{4\pi^2}\sum_{k=1}^{8}\left((K_2\ L)_k^{(sys)}\right)^2 \frac{9}{L^2}\int_0^{2L} ds\,\sqrt{\beta_x(s)\beta_y(s)}\, D_x(s)\int_0^s ds' \Big\{ \\ \qquad \sqrt{\beta_x(s')\beta_y(s')}\, D_x(s')\sin\big((\mu_x(s') \pm \mu_y(s')) - (\mu_x(s) \pm \mu_y(s))\big)\Big\}\ . \end{cases} \tag{50}$$

Terms $d^{(3)}$

The double integrals occurring in the definition of the terms $d^{(3)}$ are denoted by

$$\mathcal{I} \stackrel{\text{def}}{=} \frac{}{\ ^2}\int_0^{2L} \sqrt{\ _x\ \ _y}\ \ _x \int_0^s\ '\sqrt{\ _x\ '\ _y\ '}\ \ _x\ '\ \ (\ _x\ ' \pm\ _y\ ' -\ _x\ \pm\ _y\ )\ . \tag{51}$$

These integrals have been computed with MAPLE [8]. Their value depends on the polarity (F or D) of the central quadrupole of the cell and we report hereafter the results obtained.

$$\begin{cases} \mathcal{I}^{F,D} = \dfrac{\ ^{2}\ \ ^{4}}{1\ 0} \dfrac{\ ^{2}\ /}{\ ^{3}\ / \qquad /} \pm \dfrac{\ ^{2}\ \ ^{4}}{0} \dfrac{0 \quad ^{2}\ / \qquad ^{4}\ /}{\ ^{4}\ / \qquad /} \\ \mathcal{I}_{+}^{F,D} = -\dfrac{\ ^{2}\ \ ^{4}}{1\ 00} \dfrac{0 \quad 1 \qquad ^{2}\ / \quad 1 \qquad ^{4}\ / \qquad ^{6}\ / \qquad ^{8}\ /}{\ ^{5}\ / \qquad /} \pm \\ \qquad \dfrac{\ ^{2}\ \ ^{4}}{0} \dfrac{(1 \qquad\qquad ^{2}\ )}{\ ^{4}\ / \qquad /} \end{cases} \tag{52}$$

For $\mu \sim \pi/2$, $\alpha = 6\pi/1232$ and $L = 53.45$ m, the previous relations yield

$$\mathcal{I}^{F} = 6.34\,10^{4}\ ,\ \mathcal{I}^{D} = -5.47\,10^{4}\ \text{ and }\ \mathcal{I}_{+}^{F,D} = -1.69\,10^{5}\ . \tag{53}$$

In view of Eq. (25), the maximum contribution of the coefficients $d^{(3)}$ to the second order chromaticities $Q''_{I,II}$ is then

$$\left|\Delta^{(3)} Q''_{I,II}\right|_{max} = \frac{N_{cell}}{\pi} \left(\left(K_2\ L\right)_{max}^{(sys)}\right)^2 \left|\left(\mathcal{I}^{F} + \mathcal{I}^{D}\right) \mp \left(\mathcal{I}_{+}^{F} + \mathcal{I}_{+}^{D}\right)\right|\ , \tag{54}$$

leading to (see Tab. 4)

$$\begin{cases} \left|\Delta^{(3)} Q''_{I}\right|_{max} & = & 2.54\,10^{6} \left(\left(K_2\ L\right)_{max}^{(sys)}\right)^2 [\mathrm{m}\ \ ^{4}] & \sim & 28 \\ \left|\Delta^{(3)} Q''_{II}\right|_{max} & = & 2.41\,10^{6} \left(\left(K_2\ L\right)_{max}^{(sys)}\right)^2 [\mathrm{m}\ \ ^{4}] & \sim & 27\ , \end{cases} \tag{55}$$

which is negligible.

Terms $d^{(2)}$

After calculation of the sum appearing in the definition of $d^{(2)}$, we obtain

$$d^{(2)} = \sum_{k=1}^{8} \left(\Delta c^{(k)}\right)^2 \times \frac{\sin(N_{cell}\mu\ \ ) - N_{cell} \sin(\mu\ \ )}{2 - 2\cos(\mu\ \ )} \tag{56}$$

(see Eq. (39) for the numerical values of the coefficients $\Delta c^{(k)}$).
Since the phase advances per cell are roughly 90 in both planes, the coefficient $d_{+}^{(2)}$ can be bounded above by the following expression:

$$|d_{+}^{(2)}| \leq \frac{1}{4} \sum_{k=1}^{8} \left(\Delta c_{+}^{(k)}\right)^2 \leq 1.14\,10^{4} \left(\left(K_2\ L\right)_{max}^{(sys)}\right)^2 [\mathrm{m}\ \ ^{4}] \sim 0.13\,, \tag{57}$$

which is irrelevant in terms of Q''.
Finally, by using Eq. (31) which links the difference of phase advance per cell, $\mu\ \ = \boldsymbol{\mu_x} - \boldsymbol{\mu_y}$, and the tune split p_0, we can estimate the maximum second order chromaticities liable to be induced by the term $d^{(2)}$:

$$\begin{cases} \left(\Delta^{(2)} Q''_{I,II}\right)_{max} = \mp\pi\,(d^{(2)})_{max} = \pm 3.93\,10^{7} \left(\left(K_2\ L\right)_{max}^{(sys)}\right)^2 [\mathrm{m}\ \ ^{4}] \sim \pm 438 \text{ for } p_0 = 4 \\ \left(\Delta^{(2)} Q''_{I,II}\right)_{max} = \mp\pi\,(d^{(2)})_{max} = \pm 5.05\,10^{7} \left(\left(K_2\ L\right)_{max}^{(sys)}\right)^2 [\mathrm{m}\ \ ^{4}] \sim \pm 563 \text{ for } p_0 = 5\ , \end{cases} \tag{58}$$

This values are reached when the systematic component a_3 is the same in absolute value for each octant: $\left|\left(K_2\,L\right)_k^{(sys)}\right| = \left(K_2\,L\right)_{max}^{(sys)}$, $k = 1\cdots 8$. The effect induced in this case is then at the border of significance.

Terms $d^{(1)}$

We complete here our study by the estimate of the terms $d^{(1)}$.
A bound on the coefficient $d_+^{(1)}$ can be easily obtained showing that its contribution is negligible:

$$\begin{aligned} |d_+^{(1)}| &\stackrel{\text{def}}{=} \left| \sum_{1 \quad k<l \quad 8} \sin\!\big((\mu_{x_k}+\mu_{y_k})-(\mu_{x_l}+\mu_{y_l})\big)\, c_+^{(k)}\, c_+^{(l)} \right| \leq \sum_{1 \quad k<l \quad 8} \left|c_+^{(k)}\right| \, \left|c_+^{(l)}\right| \\ &\sim \sum_{1 \quad k<l \quad 8} \left|\Delta c_+^{(k)}\right| \, \left|\Delta c_+^{(l)}\right| \leq 1.89\,10^5 \left(\left(K_2\,L\right)_{max}^{(sys)}\right)^2 [\mathrm{m} \quad ^4] \sim 2.11 \ . \end{aligned} \tag{59}$$

Concerning the coefficient $d^{(1)}$, the problem is more delicate. We will not deal with the general case which is complicated and will simply do the following two remarks.

- Due to the LHC super-periodicity of 8 in $\mu_x - \mu_y$, the coefficient $d^{(1)}$ is given by

$$d^{(1)} = \sum_{1 \quad k<l \quad 8} \sin\left(\frac{2\pi}{8}\, p_0\,(k-l)\right) c^{(k)}\, c^{(l)} \ . \tag{60}$$

  This coefficient is then vanishing if $p_0 \equiv 0$ [4]; otherwise, the latter depends completely on the different harmonics λ_p excited by the perturbation (see Eq. (41)).

- The main point is that, unlike the coefficient c , the term $d^{(1)}$ is in general nonzero even if the harmonic λ_{p_0} is absent in the spectrum of the distribution $\left(K_2\,L\right)_k^{(sys)}$. For instance, consider the LHC with its nominal tune split $p_0 = 5$ and the three following error distributions:

$$\left\{ \begin{array}{lcll} \left(K_2\,L\right)_k^{(sys)} &=& \left(K_2\,L\right)_{max}^{(sys)} \times (-1)^k & ,\ k = 1\cdots 8 \\ \left(K_2\,L\right)_k^{(sys)} &=& \left(K_2\,L\right)_{max}^{(sys)} \times \left\{ \begin{array}{l} \cos(10\pi k/8) \\ \sin(10\pi k/8) \end{array} \right. & ,\ k = 1\cdots 8\ . \end{array} \right.$$

  As shown previously, these distributions maximise the difference coupling coefficient c in the case of an optics defined by a tune split of 4 and a tune split of 5 respectively. For the first distribution, the calculation of $d^{(1)}$ yields

$$d^{(1)} = -449.02 \ \text{ leading to } \ \Delta^{(1)}Q''_{I,II} = \mp\pi\, d^{(1)} = \pm 1411 \ , \tag{61}$$

  whereas the coefficient c is zero.
  For the second and third distribution, the contribution of $d^{(1)}$ is lower but not negligible:

$$d^{(1)} = \left\{ \begin{array}{r} 46.50 \\ -139.49 \end{array} \right. \ \text{ giving } \ \Delta^{(1)}Q''_{I,II} = \left\{ \begin{array}{l} \mp 146 \\ \pm 438 \end{array} \right. \ , \tag{62}$$

  whereas the module of the coefficient c is maximal in both cases.

2.3.5 Conclusion

In conclusion and as expected from the resonance theory, the second order chromaticities $Q''_{I,II}$ liable to be induced in the LHC by chromatic coupling phenomena are mainly due to the excitation of the difference resonance $Q_x - Q_y = p$ (compare Eq. (45) with Eqs. (58), (61) & (62)). In comparison with the *systematic* component a_3 of the dipoles, all the other sources of skew sextupolar field error are negligible in terms of Q'' (see Eq. (49) concerning the contribution of the *random* component a_3 of the dipoles and Subsection 2.3.2 for the skew sextupolar field errors expected in the other magnets of the arc and compared to the dipole ones).

Nevertheless, in the case where the difference coupling coefficient $c-$ would be zero, "by chance" and after a global correction, the contribution of the residual terms $d\ \ $ (mainly $d^{(1)}$ and $d^{(2)}$) can be non-negligible, knowing that the tolerance on Q'' is estimated to be 1000 at injection [1], this value including also the b_4 effects.

This last point imposes a first condition on the decoupling technique to be used. The cancellation of the coefficient $c\ \ $ by the correction system has to be performed in parallel with the minimisation of the non-resonant terms $d\ \ $. Moreover, the correction scheme must be constrained not to excite the sum resonance $Q_x + Q_y = p$ (i.e. the sum coupling coefficient c_+). The simplest strategy is to envisage an *arc by arc* compensation of the systematic component a_3 of the dipoles, able to cancel simultaneously the 16 coupling coefficients $c^{(k)}$:

$$\int_{k^{th}\ arc} ds\, \sqrt{\beta_x\,\beta_y}\, D_x \Big((K_2\ )_k^{(sys)} + (K_2\ )_k^{(cor)} \Big)\, e^{i(\mu_x\ \ \mu_y)} = 0 \ \ , k = 1 \cdots 8\,, \tag{63}$$

where $(K_2\ )_k^{(cor)}(s)$ denotes the corrector distribution in the k$^{\text{th}}$ arc. If such a scheme exists (which is the case as we will see in Section 3.1), the most dangerous residual term $d^{(1)}$ is automatically zero and the second order chromaticities induced by chromatic coupling are reduced to

$$Q''_{I,II} = \pi(\mp e\ \ \ + e_+)\ , \ \ \text{where}$$

$$e\ \ \ \stackrel{\text{def}}{=} \frac{1}{4\pi^2} \sum_{k=1}^{8} \int_{\substack{s' < s \\ k^{th}\ arc}} ds\, ds' \Big\{ \Big[(K_2\ )_k^{(sys)}(s) + (K_2\ )_k^{(cor)}(s) \Big] \Big[(K_2\ )_k^{(sys)}(s') + (K_2\ )_k^{(cor)}(s') \Big]$$
$$\beta_x^{\frac{1}{2}}(s)\beta_x^{\frac{1}{2}}(s')\beta_y^{\frac{1}{2}}(s)\beta_y^{\frac{1}{2}}(s')\, D_x(s)D_x(s') \sin\Big((\mu_x(s') \pm \mu_y(s')) - (\mu_x(s) \pm \mu_y(s)) \Big) \Big\}\,. \tag{64}$$

These coefficients will be found negligible for the correction scheme that we will propose in Section 3.5.

2.4 Comparison with MAD

We will close this chapter by comparing our analytical estimates with numerical results obtained with MAD and concerning the LHC injection optics version 6.-2. More precisely, we will study the chromatic coupling effects for the two following working points:

Case		Q''_I	Q''_{II}	dQ_I/dE_I	dQ_I/dE_{II}	dQ_{II}/dE_{II}
Tune split of 4	perfect machine	22.9	16.9	14.3	-1604.2	443.2
	worst case with a_3	-56733.7	56731.3	-445.0	-1212.4	-108.2
Tune split of 5	perfect machine	-49.9	5.2	-21.2	-1738.8	691.9
	worst case with a_3	-13065.0	13002.4	-242.6	-1538.1	423.5

Table 5: Second order chromaticities and anharmonicity coefficients [m 1] induced by the systematic component a_3 of the dipoles in LHC Version 6.-2; perfect machine (without a_3) and worst cases for a tune split of 4 and 5 units

- Case I. $Q_x = 63.28$ and $Q_y = 59.31$ (tune split of 4).
- Case II. $Q_x = 64.28$ and $Q_y = 59.31$ (tune split of 5).

The distributions considered for the systematic component a_3 of the dipoles will be the ones introduced in Subsection 2.3.3 (worst case):

$$a_{3_k} = a_{3_{max}} \cos(2\pi\, p_0 k/8) = \begin{cases} (-1)^{k+1}\, a_{3_{max}} & \text{for } p_0 = 4 \\ a_{3_{max}} \cos(10\pi k/8) & \text{for } p_0 = 5 \end{cases} , \; k = 1 \cdots 8 , \qquad (65)$$

with $a_{3_{max}} = 0.946$ [2].
Concerning the perfect machine (i.e without a_3), the natural chromaticity of the ring at injection is purely linear; in general, the choice adopted for the LHC is $Q' = 2$ (see Fig. 1(a) & 1(b)). On the other hand, the tune behaviour $Q_{I,II}(\delta)$ becomes strongly non-linear in presence of skew sextupole field errors in the arc dipoles (see Fig. 1(c) & 1(d)). Tab. 5 gives the values of the second order chromaticities $Q''_{I,II}$ computed with the `STATIC` command of MAD and relative to the four cases envisaged (Case I and II with and without a_3). These values are very close to the ones announced in Subsection 2.3.3 (see Eq. (45)). The small discrepancies can be explained, on the one hand, by the fact the super-periodicity of 8 in $\mu_x - \mu_y$ is not rigorously respected in the LHC, and, on the other hand, due to the fact that the relation (31) (which links the phase advance difference $\mu \; = \boldsymbol{\mu_x} - \boldsymbol{\mu_y}$ and the tune split p_0) is not entirely exact, especially for the LHC optics relative to a tune split of 4: Eq. (31) gives $\mu \; = 3.91$ and $\mu \; = 5.87$ for $p_0 = 4$ and $p_0 = 5$ respectively, to be compared with the values obtained in the real lattice $\mu \; = 4.09$ and $\mu \; = 5.89$ respectively.
Finally, as shown in Tab. 5, note that the anharmonicity induced remains relatively small in both cases compared to the one generated by the lattice sextupoles. This seems to confirm the fact that the a_3 effect is mostly chromatic and that the schemes envisaged for its correction have to work to that end.

[2] $a_{3_{max}} = 0$ 10^{-3} $^{-2}$

9712

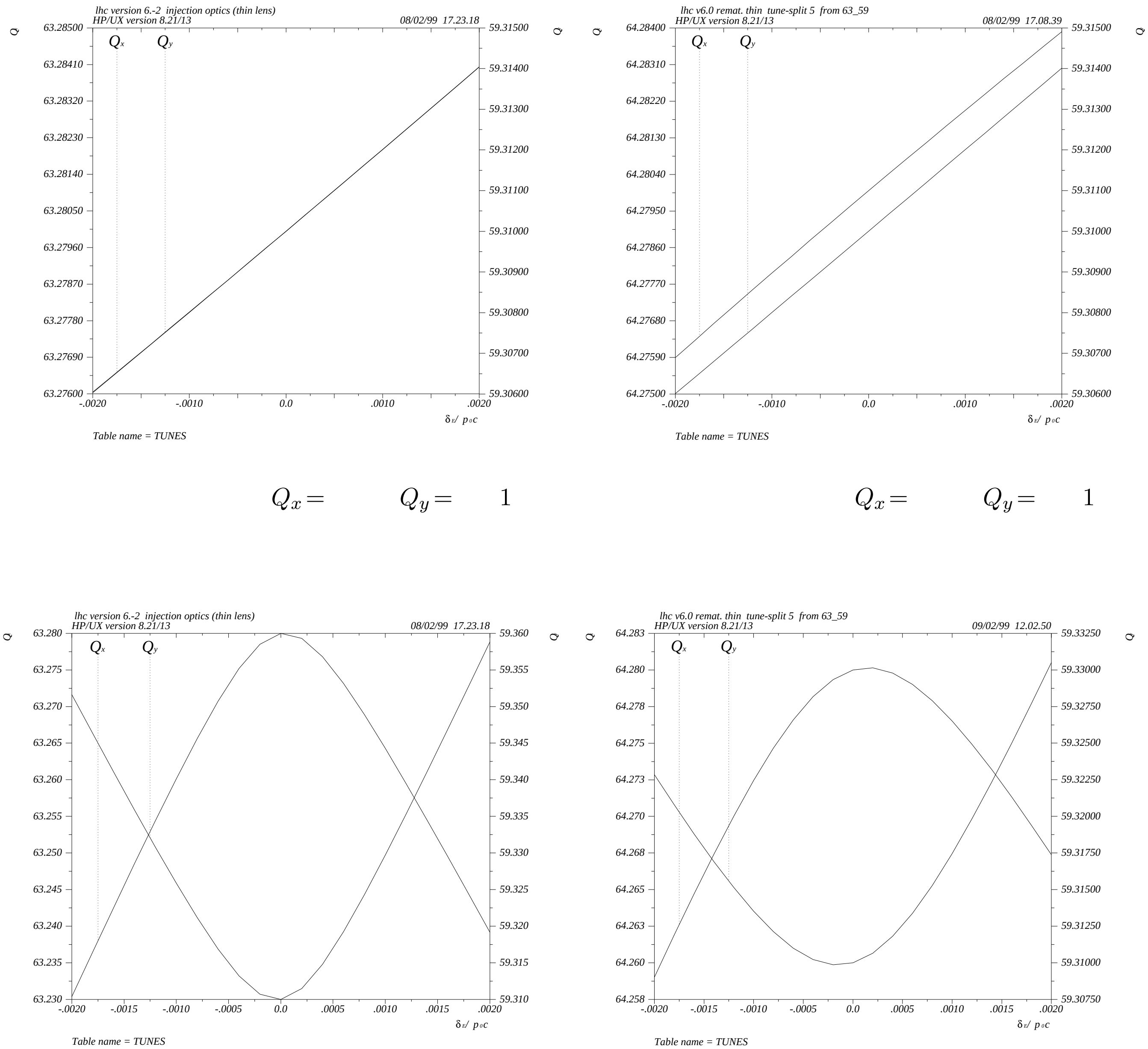


Figure 1: Dependence of the tunes $Q_{I,II}$ on the momentum deviation δ (<> ± 0.002) for a tune split of 4 and a tune split of 5 at injection: perfect machine ((a) & (b)), machine with skew sextupolar field errors in the dipoles (worst cases, (c) & (d))

3 Decoupling strategy for LHC

In view of the strong perturbation liable to be induced in the LHC by chromatic coupling, the next step is to work out an a_3 correction scheme which is both simple, of good quality, strong enough to hold up to the collision energy, and, if possible, not very costly. A solution satisfying these four conditions will be proposed in Section 3.5. Beforehand, other correction schemes, a priori possible, will be studied (Sections 3.2 to 3.4), by beginning with the one presently proposed in the LHC version 6.-2. They will be all rejected for various reasons (insufficient available gradient, geometric aberrations and then amplitude detuning induced, bad quality of the correction).

3.1 Decoupling criteria

We begin here the discussion by coming back to the decoupling condition (63) introduced in Subsection 2.3.5 (correction arc by arc of the sum and difference coupling coefficients):

$$\int_{k^{th}\ arc} ds\, \sqrt{\beta_x\,\beta_y}\, D_x \Big((K_2\ )_k^{(sys)} + (K_2\ )_k^{(cor)} \Big)\, e^{i(\mu_x \quad \mu_y)} = 0 \ \ , k = 1 \cdots 8\,.$$

In each arc, these two complex conditions are equivalent to four real conditions. Moreover, in the assumption where, for cost reasons, the correctors would be limited to one single family per arc, it can be easily checked that the condition (63) leads to the following three conditions:

(i) the corrector distribution must be symmetrical with respect to the mid-arc. In that case, the contribution to the coupling coefficients $c^{(k)}$ induced by the k^{th} corrector family is in phase with the one coming from the distribution $(K_2\ )_k^{(sys)}(s)$; as a result, two of the four initial conditions are automatically fulfilled.

(ii) the correctors must not excite the sum resonance $Q_x + Q_y = p$. Indeed, due to the uniform distribution per arc of the perturbation, we have seen in Subsection 2.3.3 that its contribution to the sum coupling coefficient c_+ was quite negligible. On the other hand, for the correctors, the distribution of which will be localised in a few cells of the arc, this contribution could be comparable to the difference coupling coefficient $c\ $ which has to be corrected. In that case, by noting $\Delta\ $ the distances to the closest sum and difference resonances, the only effect of the correction would then be to reduce the second order chromaticities[3] by the factor $\Delta_+/\Delta\ $. Knowing that $\Delta\ = -0.03$ and $\Delta_+ = -0.41$ for the nominal injection optics of the LHC, this factor corresponds roughly to one order of magnitude, which would not be sufficient in view of the tolerances and considering the Q'' values obtained in Section 2.4 ($Q''_{max} \sim 57000$ units for a tune split of 4 and $Q''_{max} \sim 13000$ units for a tune split of 5). In order to face this problem, the correctors must be arranged in pairs, each pair containing an odd number of FODO cells. In this configuration, it is

[3] $\pm$

$$Q''_{I,II} \sim -(\pm \qquad _- \ |\ _-|^2 - \qquad _+ \ |\ _+|^2) \sim \pm\frac{1}{_-}\,|\ _-|^2 \quad \frac{1}{\ }\,|\ _+|^2$$

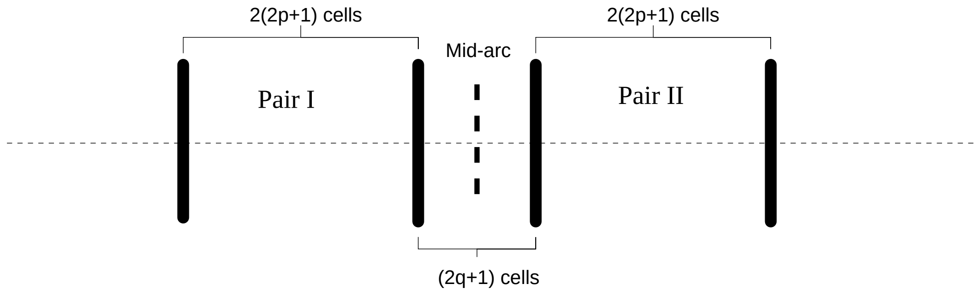


Figure 2: Optimum configuration for 4 sextupolar correctors of same polarity per arc

easy to verify that the two correctors of a same pair, spaced in betatron phase by $m\pi+\pi/2$, will act coherently on the difference resonance and will compensate each-other for the sum resonance.

(iii) the corrector strength $(K_2)_k^{(cor)}$ in the k^{th} arc is constrained by the following relation:

$$\int_{k^{th}\ arc} ds\,\sqrt{\beta_x\,\beta_y}\,D_x\Big((K_2)_k^{(sys)} + (K_2)_k^{(cor)}\Big)\,\cos(\mu_x-\mu_y) = 0\ \ , k = 1\cdots 8\,, \tag{66}$$

where the systematic skew sextupolar component $(K_2)_k^{(sys)}$ of the dipoles in the arc k will be obtained by off-line or/and on-line magnetic measurements. Note that the feed-down effects are liable to increase these values, at the very most by 10 %. Concerning the residual terms d , this (non-measurable) effect is negligible. On the other hand, an additional correction of global type (minimisation of the difference coupling coefficient c by a closest-tune approach using off-momentum buckets) will be possibly required if the a_3 induced by feed-down effect is too large.

In any case, these three conditions are similar to the ones relative to the linear coupling correction in the LHC (a_2 correction) [9]. On the other hand, insofar as the a_3 correction can be performed only with non-linear components, the strength of which is expected to be relatively large in view of the extent of the perturbation, a new condition arises:

(iv) the geometric aberrations induced by the correctors must be locally compensated. Therefore, in the case where the correctors are (skew or normal) sextupoles, they have to be arranged in pairs, each pair containing $2(2p+1)$ or $4(2q+1)$ FODO cells (which correspond to phase advances of $(2p+1)\pi$ and $(2q+1)\times 2\pi$ respectively), depending on whether the two sextupoles of a same pair have the same polarity or not. If the involved correctors are the lattice octupoles, they must be of opposite polarity in a given pair; each pair has to contain an even number of FODO cells. Finally, independently of the corrector type, we avoid interleaving these pairs.

Note that the conditions (i), (ii) and (iv) require that the number of correctors is at least equal to four in each arc of the ring. In the case where the latter are (skew or normal) sextupoles of same polarity (in a given arc), the only possible configuration is the following (see Fig. 2):

- in each arc, the correctors are arranged in two pairs, pair I and pair II, laid symmetrically with respect to the mid-arc.

• the anharmonicity (i.e. the geometric aberrations) is controlled at the level of each pair (phase advance of $(2p+1)\pi$ between two sextupoles of a same pair).
• the condition (ii) is satisfied by spacing the pairs I and II by an odd number of cells.

Finally, since the phase advance in the arc cells is not exactly 90 , this distribution has to be as much as possible confined around the mid-arc.

3.2 Preliminary scheme proposed for the a correction in LHC Version 6.-2

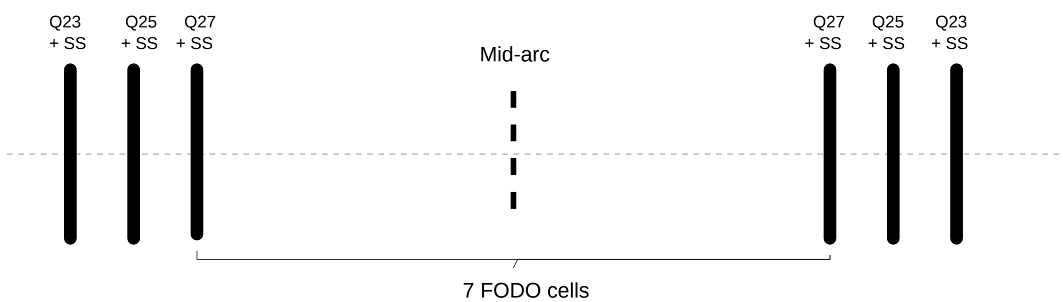


Figure 3: Preliminary a_3 correction scheme proposed in LHC Version 6.-2

Case		Q''_I	Q''_{II}	dQ_I/dE_I	dQ_I/dE_{II}	dQ_{II}/dE_{II}
Tune	worst case, no correction	-56733.7	56731.3	-445.0	-1212.4	-108.2
Split of 4	worst case with correction	-333.9	**940.7**	-1625.5	4764.3	$-\mathbf{18880.9}$
Tune	worst case, no correction	-13065.0	13002.4	-242.6	-1538.1	423.5
Split of 5	worst case with correction	-161.7	418.7	-371.7	1446.2	$-\mathbf{10052.9}$

Table 6: Second order chromaticities and anharmonicity coefficients [m ^{1}] induced by the systematic component a_3 of the dipoles, before and after correction using six 32 cm skew sextupolar correctors per arc (as proposed in LHC Version 6.-2)

Let us begin with the a_3 correction scheme presently proposed in the LHC version 6.-2. The latter consists in replacing in each arc six lattice octupoles by a family of six skew sextupoles of 32 cm each (that is a total of 96 skew sextupoles for the two rings). The conditions (i) and (ii) are respected (see Fig. 3) but the condition (iv) is not fulfilled. Moreover, the correctors are too far from the mid-arc: since the phase advance in the arc cell is not exactly $\pi/2$, the two inner correctors, although they are spaced by 7 FODO cells (see Fig. 3), are liable to excite the sum resonance $Q_x + Q_y = p$. As expected and shown in Tab. 6, this compensation scheme is not of excellent quality and induces strong geometric aberrations. Indeed, according to tracking results, satisfactory dynamic apertures are often correlated with low amplitude detuning, say lower than 10 ^{3} at $8\,\sigma$'s and for the physical emittances at injection $\epsilon_{x,y} = 7.82\,10$ ^{9} m (r.m.s). In terms of the anharmonicity coefficients given by the `STATIC` command of MAD, this criteria can be written as

$$\Delta Q_{I,II}(8\sigma) \lesssim 10^{-3} \;\Rightarrow\; \left|\frac{dQ_{I,II}}{dE_{I,II}}\right| \lesssim \frac{10^{-3}}{64 \times 7.82\,10^{-9}} \sim 2000\,\mathrm{m}^{-1}, \tag{67}$$

to be compared with the anharmonicity coefficient dQ_{II}/dE_{II} 10 or 20 times higher obtained in Tab. 6 after correction.
Finally, if the central quadrupole of the arc is x-defocusing, note that the correctors SS of this arc are placed in the vicinity of x-focusing quadrupoles where the horizontal dispersion is maximal, $D_{x_F} \sim 2.2$ m; in the contrary case, the SS's are placed in regions where the dispersion is minimal, $D_{x_D} \sim 1.1$ m, and are then two times less efficient. Thus, the condition (iii) computed with MAD (or by using the analytical expressions given in Subsection 2.3.3) leads to

$$\begin{aligned}
(K_2\,L)^{(cor)} &= \begin{cases} -15.7 \times (K_2\,L)^{(sys)}\ [\mathrm{m}^{-2}] & \text{for the SSF's} \\ -32.2 \times (K_2\,L)^{(sys)}\ [\mathrm{m}^{-2}] & \text{for the SSD's} \end{cases} \quad \text{, for a tune split of 4} \\
(K_2\,L)^{(cor)} &= \begin{cases} -14.2 \times (K_2\,L)^{(sys)}\ [\mathrm{m}^{-2}] & \text{for the SSF's} \\ -30.0 \times (K_2\,L)^{(sys)}\ [\mathrm{m}^{-2}] & \text{for the SSD's} \end{cases} \quad \text{, for a tune split of 5.}
\end{aligned} \tag{68}$$

By making the assumption that the nominal field of these correctors is $B_0/a^2 = 1500\,\mathrm{T/m}^2$ (i.e. equal to the one of the chromaticity correction sextupoles MSCH & MSCV), the SSD's could saturate around 3 TeV (2.68 TeV and 2.87 TeV for $p_0 = 4$ and $p_0 = 5$ respectively). In conclusion, this preliminary scheme seems to confirm the reliability of the different quality criteria previously introduced. In any case, if the distribution of these correctors is modified to that end and if their number is increased, a solution of this type is quite acceptable from the optics point of view.
From now on, we will pursue the idea to use the existing lattice components in order to perform the a_3 correction.

3.3 Correction attempt consisting in generating vertical orbit deviations at the level of lattice octupoles

Our first attempt consists in shifting the beam vertically in the lattice MO octupoles, if possible, in each of them and in a systematic way per arc. Even in the assumption where the present distribution of the MO's could be modified in accordance with the criteria (i), (ii) and (iv), due to the strength of the perturbation, it is easy to see that this solution is unacceptable. Indeed, the integrated maximum strength per arc of the lattice octupoles is $(K_3^+ L)_{arc} = 943.36\ \mathrm{m}^{-3}$ at injection (11 octupoles per arc, 32 cm long and of maximum gradient $B_0/a^3 = 6.7\,10^4\mathrm{T/m}^3$); that of the perturbation is $(K_2\,L)_{arc} = 138 \times 3.34\,10^{-3} = 0.46\ \mathrm{m}^{-2}$ by excluding the dipoles of the dispersion suppressors. Thus, by arguing in terms of integrated strength, from the very start of the ramp and even if the octupoles were pushed up to their maximum gradient (which is insane at injection) and used in a coherent way to act on the perturbation, the vertical offset required in the MO's would scale with the energy as $\delta y = 0.5\,\mathrm{mm} \times (\gamma/\gamma_0)$, which is unacceptable for mechanical aperture reasons.

3.4 Correction attempt consisting in generating vertical dispersion at the level of a few lattice sextupoles

An another option is to assume a non-zero vertical dispersion in the lattice sextupoles MSCH & MSCV. The non-linear correctors involved are now normal sextupoles of strength $(K_2^+)^{(cor)}$ and the condition (iii) becomes:

$$\int_{k^{th}\ arc} ds\, \sqrt{\beta_x \beta_y} \left(D_x \left(K_2\right)_k^{(sys)} + D_y \left(K_2^+\right)_k^{(cor)} \right) \cos(\mu_x - \mu_y) = 0 \ \ , k = 1 \cdots 8 \, , \qquad (69)$$

where D_y denotes the vertical dispersion induced. The latter can be generated, either thanks to the vertical correctors of the arcs, either by using the arc skew quadrupoles dedicated to the linear coupling correction.

3.4.1 Using vertical correctors

We begin with the first option. As already said, the integrated strength per arc of the perturbation is of the same order of magnitude as that of the lattice sextupoles. Consequently, the vertical dispersion induced has to be comparable to the horizontal one, i.e. $D_y \sim D_x \sim 1$ m; its sign has to be carefully controlled in order that the lattice sextupoles selected for the correction can act coherently on the perturbation. Under these conditions, simulations with MAD have shown that the use of vertical correctors to generate locally such a vertical dispersion would also induce vertical beam displacements of several tens of centimetres in the arcs, which is well beyond the mechanical aperture of the LHC.

3.4.2 Using skew quadrupoles

A much more efficient way to introduce a non-zero vertical dispersion in the arcs consists in using skew quadrupoles. These quadrupoles already exist in the LHC lattice, dedicated to the linear coupling correction. They are 32 cm long and of nominal gradient 110 T/m. Fig. (4) illustrates a possible correction scheme involving four skew quadrupoles and two or four lattice sextupoles per arc depending on whether the central quadrupole MQ of the arc considered is x-defocusing or x-focusing. For a sake of simplification, we will assume in this section that the phase advance in the arc cells is exactly 90 .

We begin by describing the distribution of skew quadrupoles. The latter are denoted by QSF and QSD on Fig. (4). In a given arc, the QSF's and QSD's are of opposite polarity; the two QSF's are spaced by π in betatron phase, idem for the two QSD's. As a result, the vertical dispersion induced remains confined inside these two pairs. Finally, since the QSF and QSD polarities are reversed, it is easy to see that the linear coupling generated locally by these skew quadrupoles is vanishing upstream and downstream from the correction station.

The lattice sextupoles used for the correction are named SF and SD on Fig. (4). In each arc, they form a new family different from the ones containing the chromatic correction sextupoles. They are assumed to have opposite polarities so that they have no effect on the linear chromaticity of the ring.

Let us study now this correction scheme in terms of the different quality criteria previously introduced.

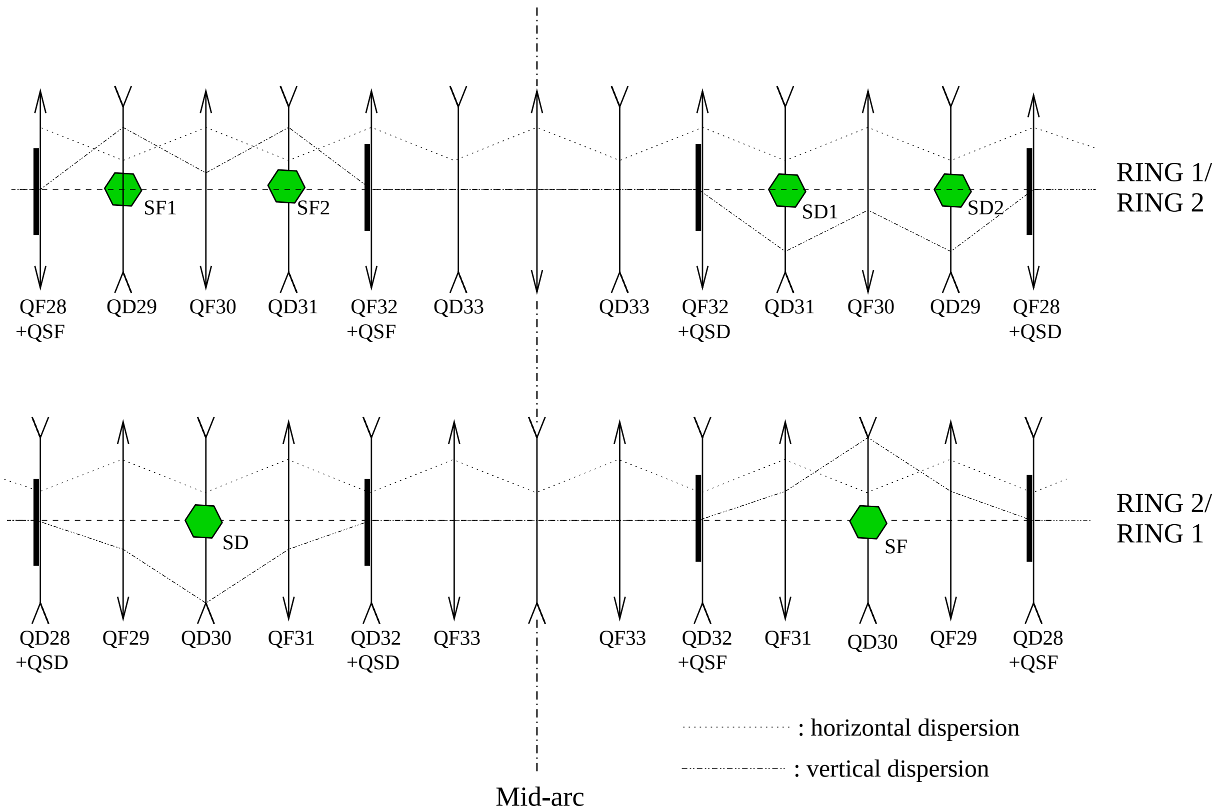


Figure 4: Correction of a_3 in Ring 1 & Ring 2 using skew quadrupoles and normal sextupoles

- The new family of sextupoles is symmetrically distributed with respect to the mid-arc. The condition (i) is fulfilled.

- If the central quadrupole of the arc is x-focusing, both pairs (SF1, SF2) and (SD1, SD2) respect the condition (ii). In the other case, for efficiency reasons, the correction station does contain only one pair (SF, SD) of sextupoles spaced by 2π in betatron phase. In brief, the criteria (ii) is fulfilled in only four arcs among the eight that the ring contains.

- The condition (iv) is always satisfied since the phase advance is exactly 2π between the sextupoles SF1 and SD1, SF2 and SD2,SF and SD, and since their respective polarities are opposite. Nevertheless, due to the interleaving of the pairs (SF1, SD1) and (SF2, SD2), small amplitude detuning effects are expected.

Strictly speaking, this solution does not satisfy the conditions (ii) and (iv) but it has the advantage of not requiring the creation of a new type of magnet. The condition (69) linking the corrector strength to the perturbation can then be computed in the following way:

- the vertical dispersion induced at the level of the pairs (SF1, SF2) and (SD1, SD2) is $D_y = \pm g_s \, D_{x_F} \, L$ where g_s $[\mathrm{m}\ ^{1}]$ denotes the integrated strength of the skew quadrupole QSF, $L = 53.45$ m is the half-cell length and $D_{x_F} \sim 2.2$ m (see Appendix A) is the horizontal dispersion in the x-focusing quadrupoles of QF type.

Within a given arc, the contribution of these correctors to the difference coupling coefficient $c\ $ is then:

$$(c\ )_1^{(cor)} = \frac{1}{2\pi} \times 4\, g_s\, g_2\, D_{x_F}\, L\, \sqrt{\beta_F \beta_D} = 5694.2\, g_s\, g_2\ , \tag{70}$$

where $\beta_D \sim 31.3\,\mathrm{m}$, $\beta_F \sim 182.5\,\mathrm{m}$ and where g_2 [m $^{\ 2}$] represents the integrated strength per sextupole.

- concerning the pair (SF, SD), a similar calculation yields:

$$(c\ )_2^{(cor)} = \frac{1}{2\pi} \times 2\, g_s\, g_2\, D_{x_D}\, \beta_F\, \sqrt{\beta_F \beta_D} = 4642.5\, g_s\, g_2\ , \tag{71}$$

  where $D_{x_D} \sim 1.1$ m is the horizontal dispersion in the x-defocusing quadrupoles of QD type. In that case, the correction station is then slightly less efficient.

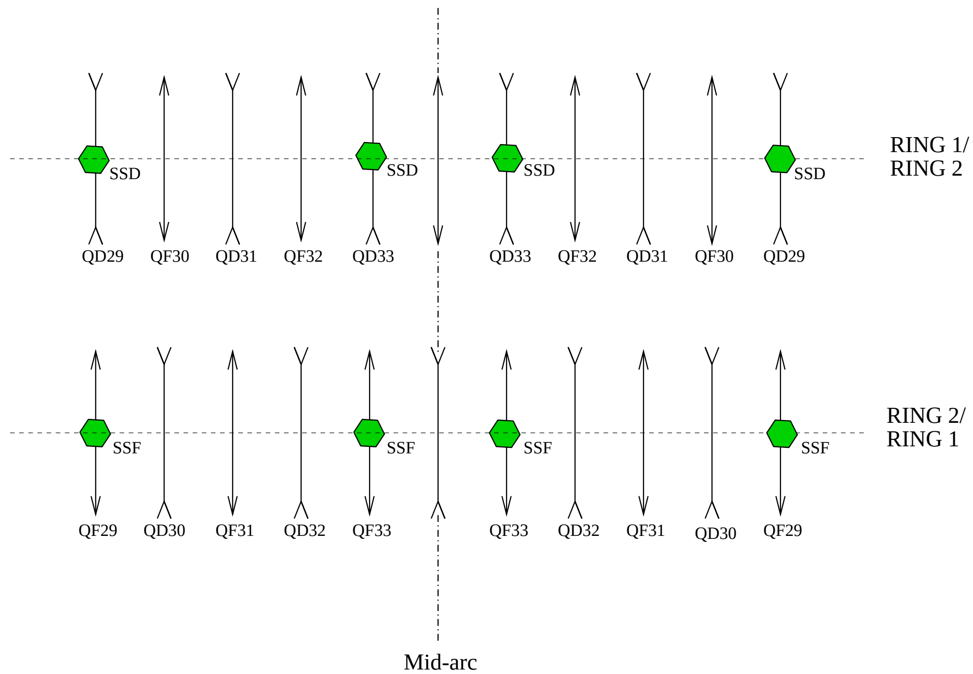
By using Subsection 2.3.3, the contribution per arc of the perturbation to the coupling coefficient $c\ $ is at the maximum 7.83 for a tune split of 4 and 6.82 for a tune split of 5. Consequently, the correctors must be able to work in the following extreme conditions:

$$(g_s\, g_2)_{max} = 1.69\, 10^{\ 3}\ \mathrm{m}^{\ 3} \text{ for } p_0 = 4 \text{ and } (g_s\, g_2)_{max} = 1.47\, 10^{\ 3}\ \mathrm{m}^{\ 3} \text{ for } p_0 = 5\ . \tag{72}$$

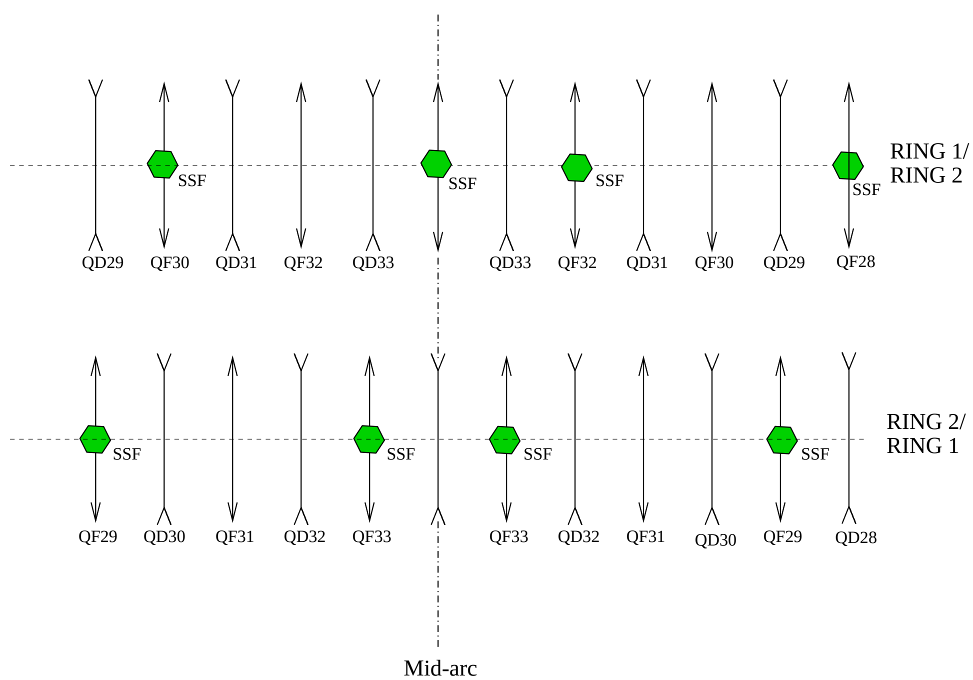
Knowing that the nominal field of the 32 cm skew quadrupoles and the one of the 1.1 m lattice sextupoles are $B_0/a = 110$ T/m and $B_0/a^2 = 1500$ T/m^2 respectively, this correction scheme could saturate around 2.5 TeV (2.49 TeV and 2.67 TeV for $p_0 = 4$ and $p_0 = 5$ respectively). Moreover, note that the vertical dispersion induced must not exceed 0.6 m at injection for reasons of mechanical aperture [10]. In the pair (SF, SD), this dispersion is maximum, equal to $\pm g_s\, D_{x_D}\, \beta_F \sim \pm 200\, g_s$. Therefore, the integrated strength of the skew quadrupoles is constrained to be less than $3\, 10^{\ 3}$ m $^{\ 1}$ at injection; in the case where the strength of the perturbation is maximal within the arc considered, this value correspond to an integrated strength per sextupole equal to ~ 0.5 m $^{\ 2}$ (see Eq. 72), which is huge at injection. Under these conditions, the number of correction stations should be doubled (or even tripled), in which case the eight skew quadrupoles per arc initially dedicated to the linear coupling compensation, would be used only for the a_3 correction!
In view of the extent of the perturbation, the use of skew sextupoles (provided they are in a sufficient number) seems then to be the only possible solution.

3.5 Proposal for a solution consisting in tilting some lattice sextupoles

The solution presented hereafter simply consists in replacing in each arc of the two rings two pairs of lattice sextupoles by two pairs of skew sextupoles of same length. By comparison with the correction scheme proposed in Section 3.2 (6 skew sextupoles per arc of 32 cm each), the total magnetic length of the a_3 correctors is multiplied in this case by a factor of 2.3, which, as we will see, will permit to achieve a full correction up to the collision energy. Moreover, the advantage of this solution lies in the fact that it does not require, as the previous one, the creation of a new type of magnet. Indeed, as for the chromatic correction



(a): F-D scheme



(b): F-F scheme

Figure 5: Two possible a_3 correction schemes using four 1.1 m skew sextupoles per arc which replace chromatic correction sextupoles

sextupoles, these skew sextupoles must be combined with an horizontal or vertical orbit corrector depending on whether they will be placed close to a QF (x-focusing quadrupole of MQ type) or close to a QD. By naming these skew sextupoles SSF and SSD respectively, it is easy to see that a chromatic correction sextupole MSCV (normal sextupole combined with a vertical corrector) becomes a skew sextupole of SSF type by a simple mechanical rotation of 90 and, in the same way, that a skew sextupole SSD can be obtained by tilting a chromaticity sextupole MSCH. Fig. 5 presents two possible correction schemes using these skew sextupoles and satisfying (almost) all the quality criteria previously introduced. The first one (F-D scheme) consists in arranging the correctors in the configuration shown in Fig. 2. In that case, 16 chromaticity sextupoles of MSCH type and 16 of MSCV type will be removed from the ring, among the 2×188 sextupoles that the latter contains. The strength of the remaining sextupoles must then be increased by roughly 9 % to ensure a chromaticity correction equivalent to the previous one. Concerning the sextupoles of MSCH type, this constraint does not cause any problem insofar as their strength is automatically two times lower than the one of the sextupoles of MSCV type (due to the ratio of 2 between the maximum and minimum dispersions, D_{x_F} and D_{x_D}, in the LHC arc cells). On the other hand, for the reasons explained hereafter, the safety margin on the MSCV strength is presently uncertain. The maximum integrated strength available per sextupole is 0.141 m ^{2} at 7 TeV (for a nominal field of 1500 T/m^2); the one of the MSCV's is 0.061 m ^{2} at collision, when the only constraint is the correction of the linear chromaticity (this value refers to the LHC collision optics version 6.-2 for $\beta^* = 0.5$ m in both low-β insertions). Nevertheless, this quite comfortable margin could be strongly reduced in the case where these sextupoles would be also used for correcting the second order chromaticity induced by the off-momentum beta-beating at collision. The study has been done on the LHC version 4.1 [3]. The strategy used was to split the sextupoles MSCV and MSCH into four independent families per ring. After the Q' and Q'' correction of for $\beta^* = 0.25$ m at IP1 and IP5, this study has shown that the strength of one of the two MSCV families was almost doubled, reaching ~ 0.12 m ^{2}; if it is always the case for the present collision optics of the LHC, this result would not refute the possibility to suppress 16 MSCV's per ring as proposed in this first correction scheme. However, a comparable study has not yet been achieved on the LHC version 6. Knowing that the efficiency of the Q'' correction (due to the off-momentum beta beating at collision) depends strongly on the tune split (which was zero in LHC Version 4), a definitive answer would be premature and certainly risky.

This problem disappears in the second scheme proposed (F-F scheme) since the latter does only require the suppression of sextupoles of MSCH type (32 per ring). In this case, the condition (i) cannot be respected in four arcs of the ring. Nevertheless, insofar as the phase advance difference per half-cell is relatively small ($(\boldsymbol{\mu_x} - \boldsymbol{\mu_y})/2 \sim$ a few degrees), this should not deteriorate the quality of the correction. The price to pay is then the addition within the lattice of two new types of short straight sections, instead of only one in the first scheme or in the one of Section 3.2 which uses the 32 cm skew sextupoles.

As shown in Tab. 7, the quality of the correction is excellent for both schemes; the anharmonicity induced is relatively small by referring to the criteria (67) introduced previously. Finally, by using MAD and the condition (iii), the perturbation and corrector strengths

Case		Q''_I	Q''_{II}	dQ_I/dE_I	dQ_I/dE_{II}	dQ_{II}/dE_{II}
F-D scheme						
Tune split of 4	Perfect machine	22.0	−18.3	33.4	−1686.6	404.2
	worst case, no correction	−56734.6	56696.1	−425.9	−1294.8	−147.2
	correction	−89.9	36.3	63.1	−1889.7	193.1
Tune split of 5	Perfect machine	−33.9	−21.4	18.0	−1773.2	682.2
	worst case, no correction	−13048.9	12975.8	−203.5	−1572.5	413.7
	correction	−79.5	19.2	33.0	−1806.0	1335.2
F-F scheme						
Tune split of 4	Perfect machine	39.6	12.3	126.9	−1644.9	451.9
	worst case, no correction	−56683.9	56693.5	−332.4	−1264.1	−99.5
	correction	−70.0	107.1	180.9	−1501.6	144.7
Tune split of 5	Perfect machine	4.9	−1.3	118.1	−1801.1	705.2
	worst case, no correction	−13003.9	12989.6	−103.3	−1602.9	436.7
	correction	−33.1	55.1	148.2	−1627.6	538.9

Table 7: Second order chromaticities and anharmonicity coefficients [m ^{1}] induced by the systematic component a_3 of the dipoles before and after correction using four 1.1 m skew sextupoles per arc

are linked by the following relations:

$$(K_2\,L)^{(cor)} \sim \begin{cases} -23 \times (K_2\,L)^{(sys)}\ [\mathrm{m}\ ^{2}] & \text{for the SSF's} \\ -46 \times (K_2\,L)^{(sys)}\ [\mathrm{m}\ ^{2}] & \text{for the SSD's} \end{cases} \text{, for a tune split of 4}$$

$$(K_2\,L)^{(cor)} \sim \begin{cases} -20 \times (K_2\,L)^{(sys)}\ [\mathrm{m}\ ^{2}] & \text{for the SSF's} \\ -41 \times (K_2\,L)^{(sys)}\ [\mathrm{m}\ ^{2}] & \text{for the SSD's} \end{cases} \text{, for a tune split of 5.} \tag{73}$$

As expected, the skew sextupoles of SSF type are two times more efficient than the ones of SSD type exclusively involved in the F-D scheme and which, according to the specification given for the perturbation strength at collision (see Tab. 4), $(K_2\,L) \sim 3.3\,10\ ^{3}\ \mathrm{m}\ ^{2}$, are just strong enough to ensure the correction up to 7 TeV (6.5 TeV for $p_0 = 4$).
In conclusion, the F-F scheme is recommended for the following two reasons:

- in this scheme, 32 chromaticity sextupoles of type MSCH are removed from the ring and replaced by skew sextupoles SSF. The gradient of the remaining 156 MSCH's will then be increased by about 20 %, which does not cause any problem insofar as their strength (constrained by the chromaticity correction) is automatically two times lower than the one of the MSCV's.

- by construction, the SSF's placed in regions where the horizontal dispersion is maximal, are two times more efficient than the skew sextupoles of SSD type. Their strength is sufficient to ensure the a_3 correction well beyond the collision energy.

4 Impact of the correction on the LHC dynamic aperture

We will finish our study by analysing the effect of the multipole a_3 on the LHC dynamic aperture and the benefits induced by its correction. All the results reported hereafter concern the *nominal injection optics of the LHC lattice version 6.-2 with a tune split of 5*, i.e $Q_x = 64.28$ and $Q_y = 59.31$.
Fig. 6 illustrates the motion of a single particle over 10'000 turns possessing the initial conditions,

$$\left\{ \begin{array}{ccccc} x & = & y & = & 12\sqrt{\beta\epsilon} \\ p_x & = & p_y & = & 0 \\ z & = & 0 & \text{and} & \delta = 7.5\,10^{\;\,4}, \end{array} \right.$$

for the three following academic cases: the perfect machine (no error), the a_3 worst case relative to a tune split of 5 (see Subsection 2.3.3) without and with correction (by the F-F scheme). Concerning the perfect machine, the smearing due to the non-zero linear chromaticity, $Q' = 2$, and to the anharmonicity induced by the chromatic correction sextupoles is relatively small (see Fig. 6(a) & 6(b)). When a_3 multipole errors are introduced in the lattice, the smearing generated by the tune ripple of average amplitude $\langle \Delta Q \rangle = Q''\delta^2/4 \sim 2\,10^{\;\,3}$ can reach several sigmas peak to peak[4] (see Fig. 6(c) & 6(d)). Finally, after compensation by the F-F correction scheme, the situation becomes quasi-identical to the one relative to the perfect machine (see Fig. 6(e) & 6(f)).
Tab. 4 gives the LHC dynamic apertures at 100'000 turns computed with SIXTRACK [11] and relative to 60 different machines. The following three cases have been considered.

- Case a): machines with field imperfections in the arc dipoles and quadrupoles (random and systematic components relative to the error table **9712**); no component a_3 (and F-F scheme not yet implemented); b_3 and b_5 correction.

- Case b): same case as Case a) but with skew sextupole field errors in the MB's and MQ's (random and systematic components).

- Case c): same case as Case b) but with a_3 compensation by the F-F correction scheme (in particular 32 chromaticity sextupoles MSCH have been removed from the lattice and replaced by skew sextupoles SSF).

The effect of the multipole a_3 is not very significant on the LHC dynamic aperture at injection. It corresponds to an average loss of $.2\,\sigma$ for the direction 15 of the phase space and $.4\,\sigma$ for the direction 45 where the resonance (0,3) excited by the skew sextupolar field errors begins to play some role. The correction is beneficial in average over the 60 seeds considered. This demonstrates, on the one hand, that the fact to remove 32 chromaticity sextupoles MSCH from the lattice does not affect the dynamic aperture of the ring; on the other hand, this shows that the geometric aberrations induced by the skew sextupolar correctors SSF remain quite acceptable for the present working point .28/.31 at injection.

[4] $Q'' \sim 1\ 000$

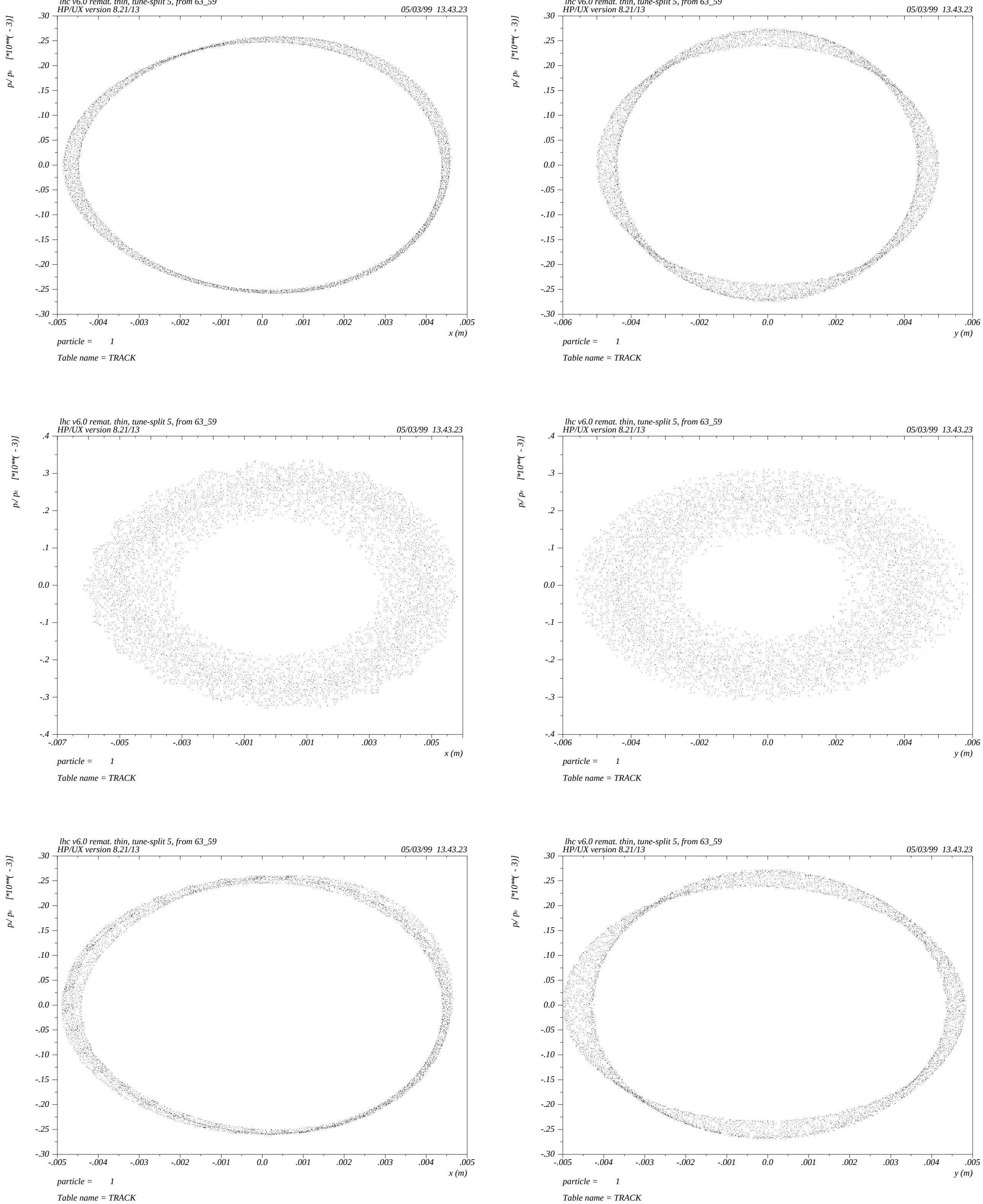


Figure 6: Smearing due to a_3. x [mm] versus p_x [mrad] and y [mm] versus p_y [mrad], turn after turn, up to 10'000 turns; one single particle with the initial conditions $x = y = 12\sqrt{\epsilon\beta}$ and $\delta p/p = 7.5\,10^{\;\,4}$ (synchrotron oscillation included): perfect machine (no errors), Fig. (a) & (b), with a_3 (worst case, no correction), Fig. (c) & (d), with a_3 correction (using the F-F scheme), Fig. (e) & (f).

Seed	Dynamic aperture [σ] at 15°			Dynamic aperture [σ] at 45°		
	Case a)	Case b)	Case c)	Case a)	Case b)	Case c)
1	12.0	11.7	11.6	13.1	12.2	12.5
2	12.4	12.3	12.1	14.0	14.4	14.2
3	11.2	11.3	12.1	14.9	14.9	14.6
4	11.9	12.5	11.8	12.1	12.8	13.0
5	11.7	11.9	12.1	12.4	13.0	12.7
6	12.4	12.1	12.7	14.4	14.5	14.4
7	11.2	11.6	11.6	13.7	12.3	12.5
8	11.9	12.8	12.1	13.3	12.8	12.4
9	13.2	13.0	13.1	13.1	12.2	14.0
10	13.5	13.0	12.9	13.5	12.6	12.7
11	12.6	12.4	12.5	12.5	12.1	12.1
12	11.3	**9.1**	11.5	13.0	14.3	13.4
13	12.8	12.6	12.9	12.9	13.1	12.5
14	12.6	12.7	12.9	**11.0**	11.5	12.4
15	11.5	11.0	11.5	14.4	14.6	14.5
16	12.6	11.7	12.1	13.7	11.6	12.4
17	10.8	11.7	**10.5**	14.7	14.6	14.5
18	13.1	13.2	**13.4**	13.0	13.5	13.2
19	12.8	11.5	12.5	13.1	14.1	14.4
20	**10.6**	12.5	12.7	15.3	14.3	14.4
21	13.0	12.4	13.0	14.9	14.1	14.8
22	12.3	12.1	13.0	15.0	15.1	**15.8**
23	12.0	11.9	12.3	12.8	13.2	12.4
24	12.2	11.6	12.2	13.3	11.6	12.3
25	11.9	10.8	11.2	11.3	11.2	**11.2**
26	**10.6**	10.9	10.7	12.7	12.9	13.0
27	13.1	12.3	13.0	13.1	12.3	12.8
28	12.4	12.9	12.8	14.7	12.9	13.8
29	11.6	11.2	11.7	13.2	14.1	12.9
30	**13.6**	12.9	13.3	15.3	15.4	15.4
31	12.8	13.2	12.9	15.0	16.3	14.6
32	13.0	12.7	12.8	14.1	14.7	14.6
33	12.4	12.8	11.0	15.4	15.6	15.7
34	12.9	11.8	12.3	14.2	13.0	13.9
35	13.0	13.1	12.7	15.7	13.8	14.1
36	12.8	12.1	13.0	13.9	14.7	14.2
37	12.0	12.3	11.9	12.0	11.6	12.1
38	13.2	11.6	12.9	14.2	13.9	14.1
39	12.5	11.5	11.9	12.6	**10.7**	11.6
40	12.2	12.6	11.8	13.3	10.9	13.7
41	12.1	11.3	12.1	14.6	12.8	13.4
42	12.9	11.5	13.0	14.5	14.7	15.0
43	13.0	12.9	13.0	12.9	12.3	12.1
44	11.3	11.0	12.3	13.6	14.2	14.6
45	11.2	12.0	12.3	14.0	12.4	14.2
46	12.6	13.0	13.0	**16.6**	**16.5**	15.1
47	12.4	12.6	11.8	14.8	15.7	14.8
48	12.9	13.1	12.9	14.0	14.0	13.4
49	12.0	12.6	12.2	12.3	11.9	12.4
50	12.2	12.7	12.3	13.4	13.0	13.3
51	11.8	11.4	11.8	12.7	12.5	13.1
52	12.3	11.9	12.5	12.9	12.1	12.6
53	12.1	11.4	12.4	14.9	13.8	14.8
54	11.2	12.0	**10.5**	11.7	**10.7**	12.1
55	12.2	12.4	11.9	13.5	13.4	14.2
56	12.8	13.4	13.0	14.9	14.4	14.3
57	11.6	12.1	12.2	13.0	11.6	12.1
58	13.2	**13.5**	12.4	11.9	12.1	13.3
59	11.4	11.7	12.1	13.9	13.9	13.3
60	12.2	11.9	12.0	14.7	14.1	14.0
Min.	10.6	9.1	10.5	11.0	10.7	11.2
Max.	13.6	13.5	13.4	16.6	16.5	15.8
Av.	12.3	12.1	12.3	13.7	13.3	13.5

Table 8: LHC injection optics Version 6.-2, tune split of 5. Dynamic aperture at 100'000 turns for two directions of the phase space, $\arctan\left(\sqrt{\epsilon_y/\epsilon_x}\right) = 15$ and 45 ; 60 seeds relative to the error table **9712** for the following cases: a) without a_3 (with b_3 and b_5 correction), b) with a_3 (with b_3 and b_5 correction), c) with a_3 (with b_3, a_3 and b_5 correction)

Conclusions

The main source of component a_3 is of geometric origin in the main dipoles (uncertainty on the systematic component per arc). Due to chromatic coupling phenomena (linear coupling proportional to $\delta p/p$) occurring in the arcs, huge second order chromaticities can be induced both at injection and collision, liable to strongly reduce the accessible momentum range required to measure $Q(\delta)$ and the dispersion. This effect would be amplified if the distance of the working point to the diagonal was reduced, which would be beneficial to minimise the strength of the high order resonances induced by the beam-beam effect in collision. Under these conditions, a dedicated correction scheme is fully justified and the F-F scheme proposed in this paper seems to be a good candidate, able to ensure a full correction of Q'' up to the ultimate energy. The 32 correctors SSF required per ring (4 per arc powered in series, see Fig. 5) take the place of 32 horizontal chromaticity sextupoles MSCH. They are skew sextupoles, 1.1 m long, combined with horizontal orbit correctors, then obtained by a simple mechanical rotation of chromaticity sextupoles of MSCV type (rotation by 90). In other words, this scheme does not imply the creation of a new type of magnet and the total number of sextupoles per ring (normal and skew) remains unchanged: $188 + 32$ (tilted) MSCV's and $188 - 32$ MSCH's.

The loss of dynamic aperture due to a_3 is not very significant (less than $.5\sigma$) and is partially reduced after correction. Nevertheless, as said before, the tune ripple due to Q'' would be increased if the working point was closer to the diagonal. In that case, the LHC dynamic aperture would be certainly more affected by the systematic component a_3 of the dipoles, which gives a second argument to justify its correction.

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A Calculation of the cell contribution to the coupling coefficients $c_{\pm}$

The purpose of this appendix is to estimate the contributions Δc of a single cell to the coupling coefficients c :

$$\Delta c \;\; = \frac{1}{2\pi} \int_{L}^{L} ds \, \sqrt{\beta_x(s)\beta_y(s)} \, D_x(s) \, K_2 \, (s) e^{i\left(\mu_x(s) \;\; \mu_y(s)\right)} \;\; ,$$

where the betatron phase origin is taken at $s = 0$. For a sake of simplification, this computation will be carried out in the thin lens approximation; in each half-cell of length L, the three dipoles will be replaced by an equivalent single dipole, the bending angle of which will be $\alpha = 6\pi/1232$; finally, the phase advances in both transverses planes will be assumed to be rigorously the same ($K_F \equiv -K_D$), say equal to $\mu = (\boldsymbol{\mu_x} + \boldsymbol{\mu_y})/2$. The integrated strength of an half-quadrupole, the curvature of both equivalent dipoles and their skew sextupolar strength will be denoted by g ($g > 0$), h and K_2 respectively, so that:

$$gL = \sin(\mu/2) \ , \ \alpha = hL \ \text{ and } \ K_2 \, L = 3 \left(K_2 \, L\right)^{(sys)} \;\; ,$$

where $\left(K_2 \, L\right)^{(sys)}$ is the integrated skew sextupolar strength for a *real dipoles* (14.3 m dipole).

Due to the symmetry with respect to the central quadrupole of the cell, the complex integrals defining the coefficients Δc reduce to two purely real integrals:

$$\Delta c \;\; = \frac{2}{2\pi} K_2 \int_0^L ds \, \sqrt{\beta_x \beta_y} \; D_x \, \cos(\mu_x \pm \mu_y) = \frac{1}{\pi} K_2 \int_0^L ds \, \left[c_x(s)c_y(s) \mp s_x(s)s_y(s)\right] D_x(s) \; .$$

In this form, the integrand appear as a sum of products containing the cosine and sine-like trajectories and the horizontal dispersion function. According to the polarity of the central quadrupole and if the focusing induced by the dipoles is neglected, this optical functions are given by

$c_x(s) = \sqrt{\beta_F}\,(1 - gs)$	$c_x(s) = \sqrt{\beta_D}\,(1 + gs)$
$c_y(s) = \sqrt{\beta_D}\,(1 + gs)$	$c_y(s) = \sqrt{\beta_F}\,(1 - gs)$
$s_x(s) = s/\sqrt{\beta_F}$	$s_x(s) = s/\sqrt{\beta_D}$
$s_y(s) = s/\sqrt{\beta_D}$	$s_y(s) = s/\sqrt{\beta_F}$
$D_x(s) = hs^2/2 + D_{x_F}(1 - gs)$	$D_x(s) = hs^2/2 + D_{x_D}(1 + gs)$
for a central quadrupole QF	for a central quadrupole QD,

$$\text{with } \; D_{x_{F,D}} = \frac{hL^2}{2\sin^2(\mu/2)}(2 \pm \sin(\mu/2)) \; \text{ and } \; \beta_{F,D} = \frac{L}{\sin(\mu/2)} \sqrt{\frac{1 \pm \sin(\mu/2)}{1 \mp \sin(\mu/2)}} \; .$$

By using MAPLE [8] to complete the calculations, we finally obtain

$$\Delta c = \frac{1}{4\pi} \frac{\alpha \, L^2}{\sin^3(\mu/2)} \Big(12 - \sin^2(\mu/2) \Big) \, \big(K_2 \, L \big)^{(sys)}$$

$$\Delta c_+ = \begin{cases} \dfrac{1}{4\pi} \dfrac{\alpha \, L^2}{\sin^3(\mu/2)} \Big(12 - 9 \sin^2(\mu/2) + 2 \sin^3(\mu/2) + 3/5 \, \sin^4(\mu/2) \Big) \, \big(K_2 \, L \big)^{(sys)} \\ \text{if the central quadrupole is x-focusing,} \\ \dfrac{1}{4\pi} \dfrac{\alpha \, L^2}{\sin^3(\mu/2)} \Big(12 - 9 \sin^2(\mu/2) - 2 \sin^3(\mu/2) + 3/5 \, \sin^4(\mu/2) \Big) \, \big(K_2 \, L \big)^{(sys)} \\ \text{if the central quadrupole is x-defocusing.} \end{cases}$$